Characteristic Functions

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Characteristic Functions

Definition

For a random variable X, the characteristic function is given by

$$\phi(t) = E(e^{itX})$$

Examples

- **Bernoulli RV:** $P(X = 1) = p$ and $P(X = 0) = 1 - p$

$$\phi(t) = 1 - p + pe^{it} = q + pe^{it}$$

- **Gaussian RV:** Let $X \sim N(\mu, \sigma^2)$

$$\phi(t) = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$$

Properties of Characteristic Functions

Theorem

If X and Y are independent, then

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(s).$$

Example (Binomial RV)

$$\phi(t) = \left(q + pe^{it}\right)^n$$

Example (Sum of Independent Gaussian RVs)

Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent. What is the distribution of $X + Y$?

Theorem

If $a, b \in \mathbb{R}$ and $Y = aX + b$, then

$$\phi_Y(t) = e^{itb}\phi_X(at).$$

Inversion and Continuity Theorems

Theorem

Random variables X and Y have the same characteristic function if and only if they have the same distribution function.

Theorem

Suppose $F_1, F_2, \ldots$ is a sequence of distribution functions with corresponding characteristic functions $\phi_1, \phi_2, \ldots$.

- *If $F_n \to F$ for some distribution function F with characteristic function ϕ, then $\phi_n(t) \to \phi(t)$ for all t.*
- *Conversely, if $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists and is continuous at $t = 0$, then ϕ is the characteristic function of some distribution function F, and $F_n \to F$.*

Questions?