# Continuous Random Variables

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# Continuous Random Variables

## Definition

A random variable is called continuous if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)\, du \;\; \text{for all} \;\; x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the probability density function of $X$.

## Example

Uniform random variable
$\Omega = [a, b]$, $X(\omega) = \omega$,

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

# Probability Density Function

- The numerical value $f(x)$ is not a probability. It can be larger than 1.
- $f(x)dx$ can be intepreted as the probability $P(x < X \leq x + dx)$ since

$$P(x < X \leq x + dx) = F(x + dx) - F(x) \approx f(x)\, dx$$

- $P(a \leq X \leq b) = \int_a^b f(x)\, dx$
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$
- $P(X = x) = 0$ for all $x \in \mathbb{R}$

# Independence

- Continuous random variables $X$ and $Y$ are independent if the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all $x$ and $y$ in $\mathbb{R}$
- If $X$ and $Y$ are independent, then the random variables $g(X)$ and $h(Y)$ are independent
- Let the joint probability distribution function of $X$ and $Y$ be $F(x, y) = P(X \leq x, Y \leq y)$.

  Then $X$ and $Y$ are said to be jointly continuous random variables with joint pdf $f_{X,Y}(x, y)$ if

  $$F(x, y) = \int_{-\infty}^{u} \int_{-\infty}^{v} f_{X,Y}(u, v)\, du\, dv$$

  for all $x, y$ in $\mathbb{R}$
- $X$ and $Y$ are independent if and only if

  $$f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y \in \mathbb{R}$$

# Expectation

- The expectation of a continuous random variable with density function $f$ is given by
$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$$
whenever this integral is finite.
- If $X$ and $g(X)$ are continuous random variables, then
$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$
- If $a, b \in \mathbb{R}$, then $E(aX + bY) = aE(X) + bE(Y)$
- If $X$ and $Y$ are independent, $E(XY) = E(X)E(Y)$
- If $k$ is a positive integer, the $k$th moment $m_k$ of $X$ is defined to be $m_k = E(X^k)$
- The $k$th central moment $\sigma_k$ is $\sigma_k = E\left[(X - m_1)^k\right]$
- The second central moment $\sigma_2 = E[(X - m_1)^2]$ is called the variance
- For a non-negative continuous RV $X$, $E(X) = \int_0^\infty [1 - F(x)]\,dx$
- Cauchy-Schwarz inequality holds for continuous random variables

# Gaussian Random Variables

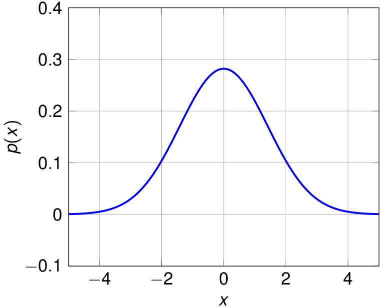
# Gaussian Random Variable

## Definition

A continuous random variable with pdf of the form

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

where $\mu$ is the mean and $\sigma^2$ is the variance.



# Notation

- $N(\mu, \sigma^2)$ denotes a Gaussian distribution with mean $\mu$ and variance $\sigma^2$
- $X \sim N(\mu, \sigma^2) \Rightarrow X$ is a Gaussian RV with mean $\mu$ and variance $\sigma^2$
- $X \sim N(0, 1)$ is termed a standard Gaussian RV

# Affine Transformations Preserve Gaussianity

## Theorem

*If $X$ is Gaussian, then $aX + b$ is Gaussian for $a, b \in \mathbb{R}, a \neq 0$.*

## Remarks

- If $X \sim N(\mu, \sigma^2)$, then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.
- If $X \sim N(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim N(0, 1)$.

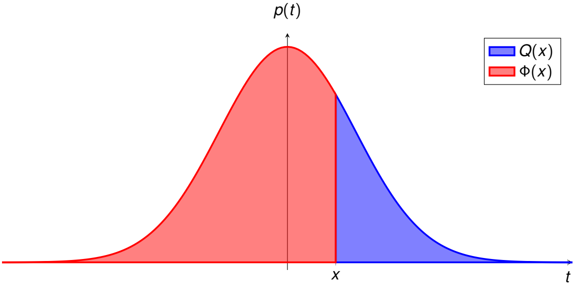
# CDF and CCDF of Standard Gaussian

- Cumulative distribution function of $X \sim N(0,1)$

$$\Phi(x) = P[X \leq x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

- Complementary cumulative distribution function of $X \sim N(0,1)$

$$Q(x) = P[X > x] = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$



# Properties of $Q(x)$

- $\Phi(x) + Q(x) = 1$
- $Q(-x) = \Phi(x) = 1 - Q(x)$
- $Q(0) = \frac{1}{2}$
- $Q(\infty) = 0$
- $Q(-\infty) = 1$
- $X \sim N(\mu, \sigma^2)$

$$P[X > \alpha] = Q\left(\frac{\alpha - \mu}{\sigma}\right)$$

$$P[X < \alpha] = Q\left(\frac{\mu - \alpha}{\sigma}\right)$$

# Jointly Gaussian Random Variables

## Definition (Jointly Gaussian RVs)

Random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if any non-trivial linear combination is a Gaussian random variable.

$$a_1 X_1 + \cdots + a_n X_n \text{ is Gaussian for all } (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \mathbf{0}$$

## Example (Not Jointly Gaussian)

$X \sim N(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim N(0, 1)$ and $X + Y$ is not Gaussian.

# Gaussian Random Vector

## Definition (Gaussian Random Vector)

A random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ whose components are jointly Gaussian.

## Notation

$\mathbf{X} \sim N(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = E[\mathbf{X}], \;\; \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T\right]$$

## Definition (Joint Gaussian Density)

If $\mathbf{C}$ is invertible, the joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right)$$

# Uncorrelated Jointly Gaussian RVs are Independent

If $X_1, \ldots, X_n$ are jointly Gaussian and pairwise uncorrelated, then they are independent.

$$
\begin{aligned}
p(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i - m_i)^2}{2\sigma_i^2}\right)
\end{aligned}
$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \mathrm{var}(X_i)$.

# Uncorrelated Gaussian RVs may not be Independent

## Example

- $X \sim N(0, 1)$
- $W$ is equally likely to be +1 or -1
- $W$ is independent of $X$
- $Y = WX$
- $Y \sim N(0, 1)$
- $X$ and $Y$ are uncorrelated
- $X$ and $Y$ are not independent

# Conditional Distribution and Density Functions

# Conditional Distribution Function

- For discrete RVs, the conditional distribution was defined as $F_{Y|X}(y|x) = P(Y \leq y|X = x)$ for any $x$ such that $P(X = x) > 0$
- For continuous RVs, $P(X = x) = 0$ for all $x$
- But considering an interval around $x$ such that $f_X(x) > 0$, we have

$$
\begin{aligned}
P(Y \leq y|x \leq X \leq x + dx) &= \frac{P(Y \leq y, x \leq X \leq x + dx)}{P(x \leq X \leq x + dx)} \\
&\approx \frac{\int_{v=-\infty}^{y} f(x, v)\, dx\, dv}{f_X(x)\, dx} \\
&= \int_{v=-\infty}^{y} \frac{f(x, v)}{f_X(x)}\, dv
\end{aligned}
$$

## Definition

The conditional distribution function of $Y$ given $X = x$ is the function $F_{Y|X}(\cdot|x)$ given by

$$F_{Y|X}(y|x) = \int_{v=-\infty}^{y} \frac{f(x, v)}{f_X(x)}\, dv$$

for any $x$ such that $f_X(x) > 0$. It is sometimes denoted by $P(Y \leq y|X = x)$.

# Conditional Density Function

## Definition

The conditional density function of $Y$ given $X = x$ is given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

for any $x$ such that $f_X(x) > 0$.

## Example (Bivariate Standard Normal Distribution)

$X$ and $Y$ are continuous random variables with joint density given by

$$f(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

where $-1 < \rho < 1$.
$[X \ Y]^T \sim N(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

What are the marginal densities of $X$ and $Y$? What is the conditional density $f_{Y|X}(y|x)$?

# Conditional Expectation

## Definition

The conditional expectation of $Y$ given $X$ is given by

$$E(Y|X = x) = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$$

## Theorem

*The conditional expectation* $\psi(X) = E(Y|X)$ *satisfies*

$$E\left[E(Y|X)\right] \quad = \quad E(Y)$$

## Example (Bivariate Standard Normal Distribution)

$X$ and $Y$ are continuous random variables with joint density given by

$$f(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}(x^2 - 2\rho xy + y^2)\right)$$

where $-1 < \rho < 1$. What is the conditional expectation of $Y$ given $X$?

# Functions of Continuous Random Variables

# Functions of a Single Random Variable

- If $X$ is a continuous random variable with density function $f$, what is the distribution function of $Y = g(X)$?

$$\begin{aligned} F_Y(y) &= P(g(X) \le y) \\ &= P\left(X \in g^{-1}(-\infty, y]\right) \\ &= \int_{g^{-1}(-\infty,y]} f(x)\, dx \end{aligned}$$

## Example (Affine transformation)

Let $X$ be a continuous random variable. What are the distribution and density functions of $aX + b$ for $a, b \in \mathbb{R}$?

## Example (Squaring a Gaussian RV)

Let $X \sim N(0, 1)$ and let $g(x) = x^2$. What are the distribution and density functions of $g(X)$?

# Functions of Two Random Variables

- Let $X_1$ and $X_2$ have the joint density function $f$. Let $Y_1 = g(X_1, X_2)$ and $Y_2 = h(X_1, X_2)$. What is the joint density function of $Y_1$ and $Y_2$?
- Let the transformation $T : (x_1, x_2) \to (y_1, y_2)$ be one-to-one. Then the transformation has an inverse $x_1 = x_1(y_1, y_2)$ and $x_2 = x_2(y_1, y_2)$ with Jacobian equal to the determinant

$$J(y_1, y_2) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \frac{\partial x_1}{\partial y_1}\frac{\partial x_2}{\partial y_2} - \frac{\partial x_1}{\partial y_2}\frac{\partial x_2}{\partial y_1}$$

- The joint density of $Y_1$ and $Y_2$ is given by

$$f_{Y_1, Y_2}(y_1, y_2) = \begin{cases} f(x_1(y_1, y_2), x_2(y_1, y_2))|J| & \text{if } (y_1, y_2) \text{ is in } T\text{'s range} \\ 0 & \text{otherwise} \end{cases}$$

## Example

Let $Y_1 = aX_1 + bX_2$ and $Y_2 = cX_1 + dX_2$ with $ad - bc \neq 0$. What is the joint density of $Y_1$ and $Y_2$?

# Sum of Continuous Random Variables

## Theorem

*If $X$ and $Y$ have a joint density function $f$, then $X + Y$ has density function*

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z - x)\, dx.$$

*If $X$ and $Y$ are independent, then*

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = \int_{-\infty}^{\infty} f_X(z - y) f_Y(y)\, dy.$$

*The density function of the sum is the convolution of the marginal density functions.*

## Example (Sum of Gaussian RVs)

Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$ be independent. What is the density function of $X + Y$?

Questions?