Discrete Random Variables

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Discrete Random Variables

Definition

A random variable is called discrete if it takes values only in some countable subset $\{x_1, x_2, x_3, \ldots\}$ of $\mathbb{R}$.

Definition

A discrete random variable X has a probability mass function $f : \mathbb{R} \to [0, 1]$ given by $f(x) = P[X = x]$

Example

- Bernoulli random variable

  $\Omega = \{0, 1\}$

$$P[X = x] = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

  where $0 \leq p \leq 1$

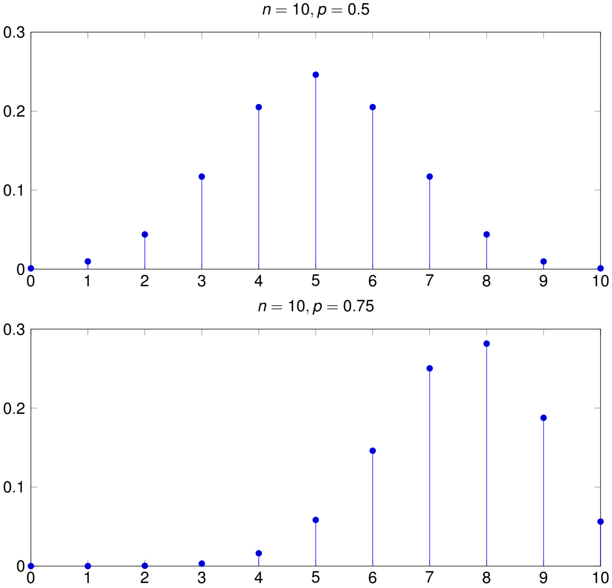
Binomial Random Variable

- An experiment is conducted n times and it succeeds each time with probability p and fails each time with probability $1 - p$
- The sample space is $\Omega = \{0, 1\}^n$ where 1 denotes success and 0 denotes failure
- Let X denote the total number of successes
- $X \in \{0, 1, 2, \ldots, n\}$
- The probability mass function of X is

$$P[X = k] = \binom{n}{k} p^k (1 - p)^{n-k} \quad \text{if } 0 \leq k \leq n$$

- X is said to have the binomial distribution with parameters n and p
- X is the sum of n Bernoulli random variables $Y_1 + Y_2 + \cdots + Y_n$

Binomial Random Variable PMF



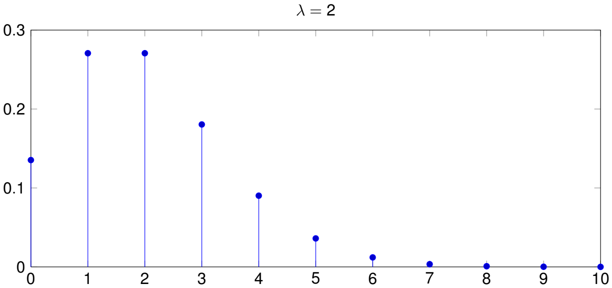
Poisson Random Variable

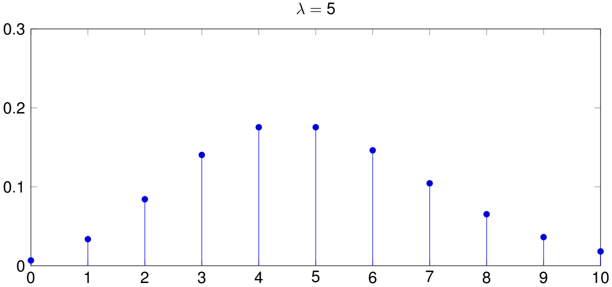
- The sample space of a Poisson random variable is $\Omega = \{0, 1, 2, 3, \ldots\}$
- The probability mass function is

$$P[X = k] = \frac{\lambda^k}{k!} e^{-\lambda} \quad k = 0, 1, 2, \ldots$$

where $\lambda > 0$

Poisson Random Variable PMF





Independence

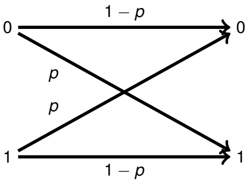
- Discrete random variables X and Y are independent if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all x and y
- A family of discrete random variables $\{X_i : i \in I\}$ is an independent family if

$$P\left(\bigcap_{i\in J}\{X_i = x_i\}\right) = \prod_{i\in J} P(X_i = x_i)$$

for all sets $\{x_i : i \in I\}$ and for all finite subsets $J \in I$

Example

Binary symmetric channel with crossover probability p



If the input is equally likely to be 0 or 1, are the input and output independent?

Consequences of Independence

- If X and Y are independent, then the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for any subsets A and B of $\mathbb{R}$
- If X and Y are independent, then for any functions $g, h : \mathbb{R} \to \mathbb{R}$ the random variables $g(X)$ and $h(Y)$ are independent
- Let X and Y be discrete random variables with probability mass functions $f_X(x)$ and $f_Y(y)$ respectively

  Let $f_{X,Y}(x, y) = P(\{X = x\} \cap \{Y = y\})$ be the joint probability mass function of X and Y

  X and Y are independent if and only if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y) \qquad \text{for all } x, y \in \mathbb{R}$$

Exercise

- Let X and Y be independent discrete random variables taking values in the positive integers
- Both of them have the same probability mass function given by

$$P[X = k] = P[Y = k] = \frac{1}{2^k} \quad \text{for } k = 1, 2, 3, \ldots$$

- Find the following
  - $P(\min\{X, Y\} \leq x)$
  - $P[X = Y]$
  - $P[X > Y]$
  - $P[X \geq kY]$ for a given positive integer k
  - $P[X \text{ divides } Y]$

Questions?