Gaussian Random Processes

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Gaussian Random Process

Definition

A random process $X(t)$ is Gaussian if its samples $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for any $n \in \mathbb{N}$ and distinct sample locations $t_1, t_2, \ldots, t_n$.

Let $\mathbf{X} = \begin{bmatrix} X(t_1) & \cdots & X(t_n) \end{bmatrix}^T$ be the vector of samples. The joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right)$$

where

$$\mathbf{m} = E[\mathbf{X}], \;\; \mathbf{C} = E\left[(\mathbf{X}-\mathbf{m})(\mathbf{X}-\mathbf{m})^T\right]$$

Properties of Gaussian Random Process

- The mean and autocorrelation functions completely characterize a Gaussian random process.
- Wide-sense stationary Gaussian processes are strictly stationary.
- If the input to a stable linear filter is a Gaussian random process, the output is also a Gaussian random process.

White Gaussian Noise

Definition

A zero mean WSS Gaussian random process with power spectral density

$$S_n(f) = \frac{N_0}{2}.$$

Properties

- Consider the output of a correlator with WGN input

  $$Z = \int_{-\infty}^{\infty} n(t)u(t)\,dt = \langle n, u \rangle$$

  where $u(t)$ is a deterministic finite-energy signal.
  Z is a Gaussian random variable with mean zero and variance $\frac{N_0}{2}\|u\|^2$
- Let $u_1(t)$ and $u_2(t)$ be linearly independent finite-energy signals. Then $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian with covariance

  $$\mathrm{cov}\left(\langle n, u_1 \rangle, \langle n, u_2 \rangle\right) = \frac{N_0}{2}\langle u_1, u_2 \rangle.$$

  If $u_1(t)$ and $u_2(t)$ are orthogonal, $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are independent.

Reference

- Chapter 3, *Fundamentals of Digital Communication*, Upamanyu Madhow, Cambridge University Press, 2008.

Questions?