Generating Random Variables

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March 8, 2013

Generating Uniform Random Variables

- $U[a, b]$ has density function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- The distribution function is

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

- When $a = 0$ and $b = 1$, we have

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

- $(b-a)U[0, 1] + a$ has the same distribution as $U[a, b]$

Generating $U[0, 1]$

- Computers can represent reals upto a finite precision
- Generate a random integer X from 0 to some positive integer m
- Generate a sample from $U[0, 1]$ as

$$U = \frac{X}{m}$$

- The linear congruential method

$$X_{n+1} = (aX_n + c) \bmod m, \quad n \geq 0$$

  where m, a, c are integers called the modulus, multiplier and increment respectively. X_0 is called the staring value.

Theorem

The linear congruential sequence has period m if and only if

- *c is relatively prime to m*
- *$b = a - 1$ is a multiple of p, for every prime p dividing m*
- *b is a multiple of 4, if m is a multiple of 4.*

Generating a Bernoulli Random Variable

- The probability mass function is given by

$$P[X = x] = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

  where $0 \leq p \leq 1$
- Generate a uniform random variable U between 0 and 1
- Generate the Bernoulli random variable by the following rule

$$X = \begin{cases} 1 & \text{if } U \leq p \\ 0 & \text{if } U > p \end{cases}$$

- How can we generate a Binomial random variable?

The Inverse Transform Method

- Suppose we want to generate a random variable with distribution function F. Assume F is one-to-one.
- Generate a uniform random variable U between 0 and 1
- $X = F^{-1}(U)$ has the distribution function F

$$P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x)$$

Example (Generating Exponential RVs)

X is an exponential RV with parameter $\lambda > 0$ if it has distribution function

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

How can it be generated?

Generating Discrete Random Variables

- Suppose we want to generate a discrete random variable X with distribution function F. F is usually not one-to-one.
- Let $x_1 \leq x_2 \leq x_3 \leq \cdots$ be the values taken by X
- Generate a uniform random variable U between 0 and 1
- Generate X according to the rule

$$X = \begin{cases} x_1 & \text{if } 0 \leq U \leq F(x_1) \\ x_k & \text{if } F(x_{k-1}) < U \leq F(x_k) \text{ for } k \geq 2 \end{cases}$$

Example (Generating Binomial RVs)

The probability mass function of a Binomial RV X with parameters n and p is

$$P[X = k] = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{if } 0 \leq k \leq n$$

How can it be generated?

Box-Muller Method for Generating Gaussian RVs

- Generate two independent uniform RVs U_1 and U_2 between 0 and 1
- Let $V_1 = 2U_1 - 1$ and $V_2 = 2U_2 - 1$
- Let $S = V_1^2 + V_2^2$.
- If $S > 1$, generate new U_i's.
- If $S \leq 1$, let
$$X_1 = V_1\sqrt{\frac{-2\ln S}{S}}, \quad X_2 = V_2\sqrt{\frac{-2\ln S}{S}}$$
- X_1 and X_2 are independent standard Gaussian random variables

$$\begin{aligned} P(X_1 \leq x_1, X_2 \leq x_2) &= \int_{\{r\cos\theta \leq x_1, r\sin\theta \leq x_2\}} \frac{1}{2\pi} e^{-\frac{r^2}{2}}\ r\, dr\, d\theta \\ &= \frac{1}{2\pi}\int_{\{x \leq x_1, y \leq x_2\}} e^{-\frac{x^2+y^2}{2}}\ dx\, dy \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_1} e^{-\frac{x^2}{2}}\ dx \cdot \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2} e^{-\frac{y^2}{2}}\ dy \end{aligned}$$

Rejection Method

- Suppose we want to generate a random variable X having density f
- Suppose X is difficult to generate using the inversion method
- Suppose there is a random variable Z which is easy to generate and which satisfies for some $a \in \mathbb{R}$

$$f(z) \leq a f_Z(z) \text{ for all } z.$$

- Generate a uniform random variable U between 0 and 1
- Generate the random variable Z
- If the event $E = \{aUf_Z(Z) \leq f(Z)\}$ occurs, set $X = Z$. Otherwise, generate another pair (U, Z) and keep trying till the event E occurs.

$$\begin{aligned}
P(Z \leq x | aUf_Z(Z) \leq f(Z)) &= \frac{P(aUf_Z(Z) \leq f(Z) \bigcap Z \leq x)}{P(aUf_Z(Z) \leq f(Z))} \\
&= \frac{\int_{-\infty}^{x} P(aUf_Z(Z) \leq f(Z) | Z = z) f_Z(z)\, dz}{\int_{-\infty}^{\infty} P(aUf_Z(Z) \leq f(Z) | Z = z) f_Z(z)\, dz} \\
&= \int_{-\infty}^{x} f(z)\, dz
\end{aligned}$$

Questions?