Hypothesis Testing

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

March 13, 2013

What is a Hypothesis?

One situation among a set of possible situations

Example (Radar)

EM waves are transmitted and the reflections observed.

Null Hypothesis Plane absent

Alternative Hypothesis Plane present

For a given set of observations, either hypothesis may be true.

What is Hypothesis Testing?

- A statistical framework for deciding which hypothesis is true
- Under each hypothesis the observations are assumed to have a known distribution
- Consider the case of two hypotheses (binary hypothesis testing)

$$\begin{aligned} H_0 &: \mathbf{Y} \sim P_0 \\ H_1 &: \mathbf{Y} \sim P_1 \end{aligned}$$

  $\mathbf{Y}$ is the random observation vector belonging to observation set $\Gamma \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}$
- The hypotheses are assumed to occur with given prior probabilities

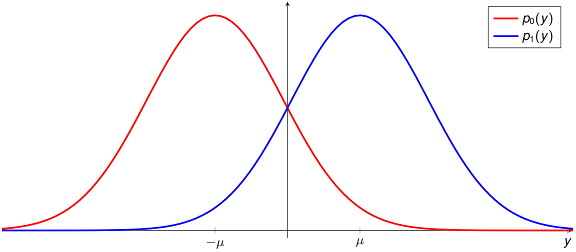
$$\begin{aligned} \Pr(H_0 \text{ is true}) &= \pi_0 \\ \Pr(H_1 \text{ is true}) &= \pi_1 \end{aligned}$$

  where $\pi_0 + \pi_1 = 1$.

Location Testing with Gaussian Error

- Let observation set $\Gamma = \mathbb{R}$ and $\mu > 0$

$$\begin{aligned} H_0 &: \quad Y \sim N(-\mu, \sigma^2) \\ H_1 &: \quad Y \sim N(\mu, \sigma^2) \end{aligned}$$



- Any point in Γ can be generated under both H_0 and H_1
- What is a good decision rule for this hypothesis testing problem which takes the prior probabilities into account?

What is a Decision Rule?

- A decision rule for binary hypothesis testing is a partition of Γ into Γ_0 and Γ_1 such that

$$\delta(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y} \in \Gamma_0 \\ 1 & \text{if } \mathbf{y} \in \Gamma_1 \end{cases}$$

  We decide H_i is true when $\delta(\mathbf{y}) = i$ for $i \in \{0, 1\}$

- For the location testing with Gaussian error problem, one possible decision rule is

$$\begin{aligned} \Gamma_0 &= (-\infty, 0] \\ \Gamma_1 &= (0, \infty) \end{aligned}$$

  and another possible decision rule is

$$\begin{aligned} \Gamma_0 &= (-\infty, -100) \cup (-50, 0) \\ \Gamma_1 &= [-100, -50] \cup [0, \infty) \end{aligned}$$

- Given that partitions of the observation set define decision rules, what is the optimal partition?

Which is the Optimal Decision Rule?

- Minimizing the probability of decision error gives the optimal decision rule
- For the binary hypothesis testing problem of H_0 versus H_1, the conditional decision error probability given H_i is true is

$$\begin{aligned} P_{e|i} &= \Pr\left[\text{Deciding } H_{1-i} \text{ is true} | H_i \text{ is true}\right] \\ &= \Pr\left[Y \in \Gamma_{1-i} | H_i\right] \\ &= 1 - \Pr\left[Y \in \Gamma_i | H_i\right] \\ &= 1 - P_{c|i} \end{aligned}$$

- Probability of decision error is

$$P_e = \pi_0 P_{e|0} + \pi_1 P_{e|1}$$

- Probability of correct decision is

$$P_c = \pi_0 P_{c|0} + \pi_1 P_{c|1} = 1 - P_e$$

Which is the Optimal Decision Rule?

- Maximizing the probability of correct decision will minimize probability of decision error
- Probability of correct decision is

$$
\begin{aligned}
P_c &= \pi_0 P_{c|0} + \pi_1 P_{c|1} \\
&= \pi_0 \int_{y \in \Gamma_0} p_0(y)\, dy + \pi_1 \int_{y \in \Gamma_1} p_1(y)\, dy
\end{aligned}
$$

- If a point y in Γ belongs to Γ_i, its contribution to P_c is proportional to $\pi_i p_i(y)$
- To maximize P_c, we choose the partition $\{\Gamma_0, \Gamma_1\}$ as

$$
\begin{aligned}
\Gamma_0 &= \{y \in \Gamma | \pi_0 p_0(y) \geq \pi_1 p_1(y)\} \\
\Gamma_1 &= \{y \in \Gamma | \pi_1 p_1(y) > \pi_0 p_0(y)\}
\end{aligned}
$$

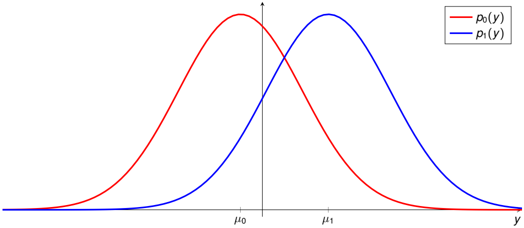
- The points y for which $\pi_0 p_0(y) = \pi_1 p_1(y)$ can be in either Γ_0 and Γ_1 (the optimal decision rule is not unique)

Location Testing with Gaussian Error

- Let $\mu_1 > \mu_0$ and $\pi_0 = \pi_1 = \frac{1}{2}$

$$\begin{aligned} H_0 &: \quad Y = \mu_0 + Z \\ H_1 &: \quad Y = \mu_1 + Z \end{aligned}$$

where $Z \sim N(0, \sigma^2)$



$$\begin{aligned} p_0(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_0)^2}{2\sigma^2}} \\ p_1(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_1)^2}{2\sigma^2}} \end{aligned}$$

Location Testing with Gaussian Error

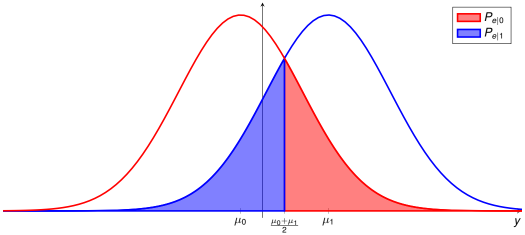
- Optimal decision rule is given by the partition $\{\Gamma_0, \Gamma_1\}$

$$\begin{aligned}\Gamma_0 &= \{y \in \Gamma | \pi_0 p_0(y) \geq \pi_1 p_1(y)\} \\ \Gamma_1 &= \{y \in \Gamma | \pi_1 p_1(y) > \pi_0 p_0(y)\}\end{aligned}$$

- For $\pi_0 = \pi_1 = \frac{1}{2}$

$$\begin{aligned}\Gamma_0 &= \left\{ y \in \Gamma \middle| y \leq \frac{\mu_1 + \mu_0}{2} \right\} \\ \Gamma_1 &= \left\{ y \in \Gamma \middle| y > \frac{\mu_1 + \mu_0}{2} \right\}\end{aligned}$$

Location Testing with Gaussian Error



$$P_{e|0} = \Pr\left[Y > \frac{\mu_0+\mu_1}{2} \middle| H_0\right] = Q\left(\frac{\mu_1-\mu_0}{2\sigma}\right)$$

$$P_{e|1} = \Pr\left[Y \le \frac{\mu_0+\mu_1}{2} \middle| H_1\right] = \Phi\left(\frac{\mu_0-\mu_1}{2\sigma}\right) = Q\left(\frac{\mu_1-\mu_0}{2\sigma}\right)$$

$$P_e = \pi_0 P_{e|0} + \pi_1 P_{e|1} = Q\left(\frac{\mu_1-\mu_0}{2\sigma}\right)$$

This P_e is for $\pi_0 = \pi_1 = \frac{1}{2}$

Location Testing with Gaussian Error

- Suppose $\pi_0 \neq \pi_1$
- Optimal decision rule is still given by the partition $\{\Gamma_0, \Gamma_1\}$

$$
\begin{aligned}
\Gamma_0 &= \{y \in \Gamma | \pi_0 p_0(y) \geq \pi_1 p_1(y)\} \\
\Gamma_1 &= \{y \in \Gamma | \pi_1 p_1(y) > \pi_0 p_0(y)\}
\end{aligned}
$$

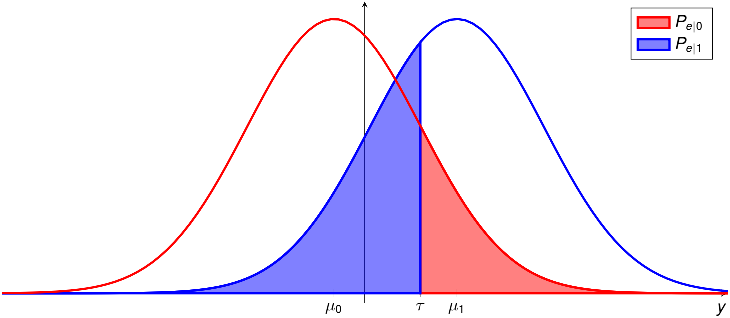
- The partitions specialized to this problem are

$$
\begin{aligned}
\Gamma_0 &= \left\{ y \in \Gamma \middle| y \leq \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} \right\} \\
\Gamma_1 &= \left\{ y \in \Gamma \middle| y > \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} \right\}
\end{aligned}
$$

Location Testing with Gaussian Error

Suppose $\pi_0 = 0.6$ and $\pi_1 = 0.4$

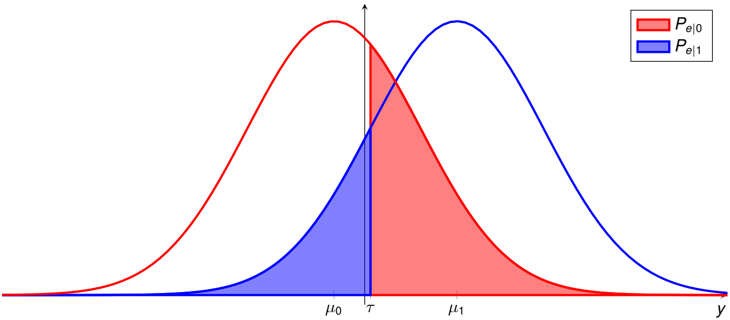
$$\tau = \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} = \frac{\mu_1 + \mu_0}{2} + \frac{0.4054\sigma^2}{(\mu_1 - \mu_0)}$$



Location Testing with Gaussian Error

Suppose $\pi_0 = 0.4$ and $\pi_1 = 0.6$

$$\tau = \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} = \frac{\mu_1 + \mu_0}{2} - \frac{0.4054\sigma^2}{(\mu_1 - \mu_0)}$$



M-ary Hypothesis Testing

- M hypotheses with prior probabilities π_i, $i = 1, \ldots, M$

$$\begin{aligned} H_1 &: \mathbf{Y} \sim P_1 \\ H_2 &: \mathbf{Y} \sim P_2 \\ &\vdots \\ H_M &: \mathbf{Y} \sim P_M \end{aligned}$$

- A decision rule for M-ary hypothesis testing is a partition of Γ into M disjoint regions $\{\Gamma_i | i = 1, \ldots, M\}$ such that

$$\delta(\mathbf{y}) = i \text{ if } \mathbf{y} \in \Gamma_i$$

We decide H_i is true when $\delta(\mathbf{y}) = i$ for $i \in \{1, \ldots, M\}$

- Minimum probability of error rule is

$$\delta_{\text{MPE}}(\mathbf{y}) = \arg \max_{1 \leq i \leq M} \pi_i p_i(\mathbf{y})$$

Maximum A Posteriori Decision Rule

- The a posteriori probability of H_i being true given observation $\mathbf{y}$ is

$$P\left[H_i \text{ is true} \middle| \mathbf{y}\right] = \frac{\pi_i p_i(\mathbf{y})}{p(\mathbf{y})}$$

- The MAP decision rule is given by

$$\delta_{\text{MAP}}(\mathbf{y}) = \arg \max_{1 \le i \le M} P\left[H_i \text{ is true} \middle| \mathbf{y}\right] = \delta_{\text{MPE}}(\mathbf{y})$$

MAP decision rule = MPE decision rule

Maximum Likelihood Decision Rule

- The ML decision rule is given by

$$\delta_{\text{ML}}(\mathbf{y}) = \arg\max_{1 \le i \le M} p_i(\mathbf{y})$$

- If the M hypotheses are equally likely, $\pi_i = \frac{1}{M}$
- The MPE decision rule is then given by

$$\delta_{\text{MPE}}(\mathbf{y}) = \arg\max_{1 \le i \le M} \pi_i p_i(\mathbf{y}) = \delta_{\text{ML}}(\mathbf{y})$$

For equal priors, ML decision rule = MPE decision rule

Questions?