Probability Spaces

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

January 9, 2013

Probability Theory

- Branch of mathematics which pertains to random phenomena
- Used to model uncertainty in the real world
- Applications
  - Statistical Inference
  - Communications
  - Signal Processing
  - Algorithms
  - Finance
  - Gambling

What is Probability?

- Classical definition: Ratio of outcomes favorable to an event to the total number of outcomes provided all outcomes are equally likely.

$$P(A) = \frac{N_A}{N}$$

- Relative frequency definition:

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}$$

- Axiomatic definition: A countably additive function defined on the set of events with range in the interval $[0, 1]$.
- The axiomatic definition will be used in this course

Sample Space

Definition

The set of all possible outcomes of an experiment is called the sample space and is denoted by Ω.

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Tossing of two coins: $\Omega = \{(H, H), (T, H), (H, T), (T, T)\}$
- A box contains three balls: one red, one green and one blue. Ball is drawn, replaced and a ball is drawn again. What is Ω? Without the replacement, what is Ω?
- Coin is tossed until heads appear. What is Ω?
- Life expectancy of a random person. $\Omega = [0, 120]$ years

Events

- An event is a subset of the sample space

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$.
  $E = \{\text{Heads}\}$ is the event that a head appears on the flip of a coin.
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
  $E = \{2, 4, 6\}$ is the event that an even number appears.
- Life expectancy. $\Omega = [0, 120]$.
  $E = [50, 120]$ is the event that a random person lives beyond 50 years.

- Can all subsets of a sample space be events?
  - Yes, if the sample space is finite or countable
  - No, if the sample space is uncountable

Which subsets must be events?

Let $\mathcal{F}$ be a subset of 2^Ω consisting of all events.

- If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$ and $A \cap B \in \mathcal{F}$
- If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$
- $\Omega \in \mathcal{F}$

The above requirements imply

- $\phi \in \mathcal{F}$
- If $A_1, \ldots, A_n \in \mathcal{F}$, then $\bigcup_{i=1}^n A_i \in \mathcal{F}$

To deal with infinite sample spaces, $\mathcal{F}$ needs to be a σ-field

σ-fields

Definition

A collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies

(a) $\phi \in \mathcal{F}$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(c) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Examples

- $\mathcal{F} = \{\phi, \Omega\}$ is the smallest σ-field
- If $A \subseteq \Omega$, $\mathcal{F} = \{\phi, A, A^c, \Omega\}$ is a σ-field
- 2^{Ω} is a σ-field

Probability Measure

Definition

A probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ satisfying

(a) $P(\phi) = 0$, $P(\Omega) = 1$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Examples

- Coin toss: $\Omega = \{\text{H}, \text{T}\}$, $\mathcal{F} = \{\phi, \text{H}, \text{T}, \Omega\}$

$$P(\phi) = 0, \quad P(\text{H}) = p, \quad P(\text{T}) = 1 - p, \quad P(\Omega) = 1$$

- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$

$$P(A) = \sum_{i \in A} p_i \text{ for any } A \subseteq \Omega$$

Probability Space

Definition

A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω and a probability measure P on $(\Omega, \mathcal{F})$.

Questions?