Properties of Probability Spaces

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Probability Space

Definition

A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω and a probability measure P on $(\Omega, \mathcal{F})$.

Definition

A probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ satisfying

(a) $P(\phi) = 0$, $P(\Omega) = 1$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Some Properties of Probability Spaces

- $P(A^c) = 1 - P(A)$
- If $A \subseteq B$, then $P(A) \leq P(B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots A_n)$$
- $$P\left(\bigcap_{i=1}^{n} A_i\right) = \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cup A_j) + \sum_{i<j<k} P(A_i \cup A_j \cup A_k) - \cdots + (-1)^{n+1} P(A_1 \cup A_2 \cup \cdots A_n)$$

P is a continuous set function

Theorem

Let $A_1, A_2, \ldots$ be an increasing sequence of events, so that $A_1 \subseteq A_2 \subseteq \cdots$. Let A be their limit

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i.$$

Then $P(A) = \lim_{i \to \infty} P(A_i)$.
Similarly, if $B_1, B_2, \ldots$ be a decreasing sequence of events, so that $B_1 \supseteq B_2 \supseteq \cdots$. Let B be their limit

$$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i.$$

Then $P(B) = \lim_{i \to \infty} P(B_i)$.

Questions?