Random Variables

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Measurements in Experiments

- In many experiments, we are interested in some real-valued measurement

Example

- A coin is tossed twice. We want to count the number of heads which appear.
  $\Omega = \{HH, HT, TH, TT\}$
  Let $X(\omega)$ be the number of heads for $\omega \in \Omega$.
  $X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0$

- We are also interested in knowing which measurements are more likely and which are less likely
- The distribution function $F : \mathbb{R} \to [0, 1]$ captures this information where

$$\begin{aligned} F(x) &= \text{Probability that } X(\omega) \text{ is less than or equal to } x \\ &= P\left(\{\omega \in \Omega : X(\omega) \leq x\}\right) \end{aligned}$$

- Is $\{\omega \in \Omega : X(\omega) \leq x\}$ always an event? Does it always belong to the σ-field $\mathcal{F}$ of the experiment?

Random Variables

Definition (Random Variable)

A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$.

Definition (Distribution Function)

The distribution function of a random variable X is the function $F : \mathbb{R} \to [0, 1]$ given by $F(x) = P(X \leq x)$

Examples

- Counting heads in two tosses of a coin.
- Constant random variable

$$X(w) = c \;\text{ for all }\; \omega \in \Omega$$

- Bernoulli random variable

Properties of the Distribution Function

- $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$
- If $x < y$, then $F(x) \leq F(y)$
- F is right continuous, $F(x + h) \to F(x)$ as $h \downarrow 0$

Discrete Random Variables

Definition

A random variable is called discrete if it takes values only in some countable subset $\{x_1, x_2, x_3, \ldots\}$ of $\mathbb{R}$.

Definition

A discrete random variable X has a probability mass function $f : \mathbb{R} \to [0, 1]$ given by $f(x) = P[X = x]$

Examples

- Counting heads in two tosses of a coin.
- Constant random variable

$$X(w) = c \text{ for all } \omega \in \Omega$$

- Bernoulli random variable

Continuous Random Variables

Definition

A random variable is called continuous if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u)\, du \;\text{ for all }\; x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the probability density function of X.

Example

- Uniform random variable
  $\Omega = [a, b]$, $X(\omega) = \omega$, $f(x) = \frac{1}{b-a}$
- Gaussian random variable
  $\Omega = \mathbb{R}$, $X(\omega) = \omega$, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Random Vectors

Definition

A random vector is a vector of random variables

Definition

The joint distribution function of a random vector $\mathbf{X} = (X_1, \dots, X_n)$ is the function $F_{\mathbf{X}} : \mathbb{R}^n \to [0, 1]$ given by $F_{\mathbf{X}}(\mathbf{x}) = P(\mathbf{X} \le \mathbf{x})$

Properties of the Joint Distribution Function

- $\lim_{x \to -\infty, y \to -\infty} F_{X,Y}(x, y) = 0$, $\lim_{x \to \infty, y \to \infty} F_{X,Y}(x, y) = 1$
- If $x_1 \le x_2, y_1 \le y_2$, then $F(x_1, y_1) \le F(x_2, y_2)$
- F is continuous from above

$$F_{X,Y}(x + u, y + v) \to F(x, y) \text{ as } u, v \downarrow 0$$

- $\lim_{y \to \infty} F_{X,Y}(x, y) = F_X(x)$, $\lim_{x \to \infty} F_{X,Y}(x, y) = F_Y(y)$

Questions?