Sampling Models

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Equally Likely Outcomes in a Finite Sample Space

- Many interesting experiments have a finite sample space and equally likely outcomes
- The σ-field in this case is the power set of the sample space
- The probability of an event is the ratio of the cardinalities of the event and the sample space

  $$P(A) = \frac{|A|}{|\Omega|}$$

  for $A \subseteq \Omega$
- In this situation, to find the probability of an event A we need to find its cardinality i.e. count the number of elements in it

Fundamental Rule

A number of multiple choices are to be made. There are m_1 possibilities for the first choice, m_2 for the second, m_3 for the third and so on. If these choices can be combined freely, then the total number of possibilities for the whole set of choices is equal to

$$m_1 \times m_2 \times m_3 \times \cdots .$$

Examples

- A man has three shirts and two ties. How many ways can he dress up?
- A man always eats a 4-course dinner consisting of a beverage, starter, main course and dessert. If a restaurant serves 4 beverages, 2 soups, 3 main courses and 4 desserts, how many possible different dinners can he have?
- In how many ways can five dice appear when they are rolled?
- In how many ways can five dice show different faces when they are rolled?

Sampling Models

- An urn contains m distinguishable balls numbered 1 to m
- n balls will be drawn under various specified conditions
- Number of all possible outcomes will be counted in each case

Sampling with Replacement and with Ordering

- The n balls are drawn sequentially, each drawn ball being put back into the urn before the next drawing
- The numbers on the balls are recorded in the order of their appearance
- One outcome of this experiment can be described by an n-tuple $(a_1, a_2, \ldots, a_n)$ where each a_i can be any integer from 1 to m
- By the fundamental rule, the total number of outcomes is

$$\underbrace{m \times m \times \cdots \times m}_{n \text{ times}} = m^n$$

Sampling without Replacement and with Ordering

- The n balls are drawn sequentially, each drawn ball is left out of the urn for the next drawing
- The numbers on the balls are recorded in the order of their appearance
- One outcome of this experiment can be described by an n-tuple $(a_1, a_2, \ldots, a_n)$ where the a_i's are all different integers between 1 and m
- By the fundamental rule, the total number of outcomes is

$$m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)$$

Permutations of m Distinguishable Balls

- This is equivalent to sampling all the balls in the urn without replacement and with ordering
- By the fundamental rule, the total number of outcomes is

$$m! = m \cdot (m-1) \cdot (m-2) \cdots 2 \cdot 1$$

Sampling without Replacement and without Ordering

- The n balls are drawn sequentially, each drawn ball is left out of the urn for the next drawing
- The order of appearance of the balls is not recorded. This is equivalent to grabbing n balls in one go.
- One outcome of this experiment can be described by a set $\{a_1, a_2, \ldots, a_n\}$ where the a_i's are all different integers between 1 and m
- Let x be the total number of outcomes
- By permuting each group of n balls, we can get any n-tuple which contains these n balls
- Thus $x \cdot n! = m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)$
- The total number of outcomes is

$$x = \frac{m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)}{n!} = \frac{m!}{n!(m-n)!} = \binom{m}{n}$$

- The number $\binom{m}{n}$ is called a binomial coefficient

Permutations of m Balls Distinguishable by Groups

- Suppose there are m_1 balls of color 1, m_2 balls of color 2, ..., m_r balls of color r such that
$$m_1 + m_2 + \cdots + m_r = m$$
- The colors are distinguishable but the balls of the same color are not distinguishable
- How many distinguishable arrangements of these m balls are possible?
- Let x be the total number of arrangements
- Then
$$x \cdot m_1!m_2!\cdots m_r! = m! \Rightarrow x = \frac{m!}{m_1!m_2!\cdots m_r!}$$
- The number x is called a multinomial coefficient

Sampling with Replacement and without Ordering

- The n balls are drawn sequentially, each drawn ball being put back into the urn before the next drawing
- The order of appearance of the balls is not recorded
- One outcome of this experiment can be described by an m-tuple $(a_1, a_2, \ldots, a_m)$ where each a_i can be any integer from 0 to n and $\sum_{i=1}^{m} a_i = n$
- The total number of outcomes is

$$\binom{m+n-1}{n}$$

Partition of m Balls into r Groups

- Suppose we want to divide m distinguishable balls into r groups; m_1 in the first group, m_2 in the second group, ... , m_r in the rth group

$$m_1 + m_2 + \cdots + m_r = m$$

- The ordering within a group does not matter
- The number of ways of choosing m_1 balls from m balls ignoring order is

$$\binom{m}{m_1}$$

- The number of ways of choosing m_2 balls from $m - m_1$ balls ignoring order is

$$\binom{m - m_1}{m_2}$$

- By the fundamental rule, the total number of outcomes is

$$\binom{m}{m_1} \cdot \binom{m - m_1}{m_2} \cdots \binom{m - m_1 - m_2 - \cdots - m_{r-1}}{m_r} = \frac{m!}{m_1!m_2! \cdots m_r!}$$

Examples

- A deck of 52 cards is shuffled. What is the probability that the four aces are found in a row?
- Six dice are rolled. What is the probability of getting three pairs?
- What is the probability that among n people there are at least two who have the same birthday? Ignore leap years.

Reference

- Chapter 3, *Elementary Probability Theory*, K. L. Chung and F. AitSahlia, 2002 (4th Edition)

Questions?