EE 325: Probability and Random Processes (Spring 2014)
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Practice Problem Set 4 — February 17, 2014

The abbreviation GS X.Y.Z denotes that the problem is from Grimmett and Stirzaker's book (Chapter X, Section Y, Problem Z). The solutions to such problems can be found in the companion volume *One Thousand Exercises in Probability* by the same authors.

1. (GS 3.7.6) Let $X_1, X_2, \ldots$ be identically distributed random variables with mean μ, and let N be a random variable taking values in the non-negative integers and independent of the X_i's. Let $S = X_1 + X_2 + \cdots + X_N$. Show that $E(S|N) = \mu N$, and deduce that $E(S) = \mu E(N)$.

2. (GS 3.8.1) Let X and Y be independent variables, X being equally likely to take any value in $\{0, 1, \ldots, m\}$ and Y being equally likely to take any value in $\{0, 1, 2, \ldots, n\}$. Find the probability mass function of $X + Y$.

3. (GS 3.11.6) Let X and Y be independent Poisson variables with parameters λ and μ respectively. Show that
   (a) $X + Y$ is a Poisson random variable with parameter $\lambda + \mu$.
   (b) the conditional distribution of X, given $X + Y = n$ is binomial. Find the parameters of the binomial distribution.

4. (GS 3.11.7) If X is a geometric random variable, show that $P(X = n + k|X > n) = P(X = k)$ for $k, n \geq 1$. Why do you think this is called the 'lack of memory' property?

5. (GS 3.11.8) Show that the sum of two independent binomial random variables, $\text{bin}(m, p)$ and $\text{bin}(n, p)$ respectively, is $\text{bin}(m + n, p)$. The notation $\text{bin}(n, p)$ represents a binomial random variable with parameters n and p.

6. (GS 3.11.12) Suppose X and Y take values in $\{0, 1\}$ with joint probability mass function $f(x, y)$. Let $f(0, 0) = a, f(0, 1) = b, f(1, 0) = c$ and $f(1, 1) = d$. Find necessary and sufficient conditions for X and Y to be:
   (a) uncorrelated
   (b) independent

7. (GS 3.11.14) Let $X_1, X_2, \ldots, X_n$ be independent random variables, and suppose X_k is a Bernoulli random variable with parameter p_k. Show that $Y = X_1 + X_2 + \cdots + X_n$ has mean and variance given by
$$E(Y) = \sum_{k=1}^{n} p_k, \qquad \text{var}(Y) = \sum_{k=1}^{n} p_k(1 - p_k).$$

8. (GS 4.2.1) I am selling my house, and have decided to accept the first offer exceeding $\$K$. Assuming that the offers are independent random variables with common distribution function F, find the expected number of offers received before I sell my house.

9. (GS 4.3.2) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables for which $E(X_1^{-1})$ exists. Show that, if $m \leq n$, then $E\left(\frac{S_m}{S_n}\right) = \frac{m}{n}$, where $S_m = X_1 + X_2 + \cdots + X_m$.