EE 325: Probability and Random Processes (Spring 2014)
Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay

Midsemester Exam : 30 points — February 21, 2014

1. (2 points) A student appears for four exams in succession. The probability of his passing the first exam is p. The probability of his passing each succeeding exam is p or $\frac{p}{2}$ depending on whether he passes or fails the preceding one. What is the probability he passes at least three exams?

2. (2 points) A man has five coins: two of them have heads on both sides, one has tails on both sides and two are normal.
   (a) He shuts his eyes, picks a coin at random and tosses it. He opens his eyes and sees that the coin is showing heads. What is the probability that the lower face of the coin is a heads?
   (b) He shuts his eyes again and tosses the coin again. He opens his eyes and sees that the coin is showing heads. What is the probability that the lower face of the coin is a heads?

3. (a) (1 point) A biased coin has a probability p of showing heads when tossed. In n independent tosses of the coin, what is the probability that the kth heads appears on the nth toss?
   (b) (1 point) A fair coin is tossed n times each by two people. What is the probability that they get the same number of heads?

4. (2 points) Suppose $A_i, 1 \leq i \leq 5$, are independent events. Show that
   (a) $(A_1 \cup A_2) \cap A_3$ and $A_4^c \cup A_5^c$ are independent.
   (b) $(A_1 \cup A_2)$, A_3 and A_5^c are independent. (Note: There are three events here)

5. (2 points) Using the clues given below, fill in the missing entries in the joint probability mass function of X and Y.

Y/X	1	2	3
1	?	?	?
2	?	0	?
3	0	?	0

Table 1: Joint probability mass function $f_{X,Y}(x, y)$

   For $k = 1, 2, 3$,
   - $P(Y = 1|X = k) = \frac{2}{3}$
   - $P(X = k|Y = 1) = \frac{k}{6}$

6. (2 points) Two players A and B alternately toss a coin which shows heads with probability p. The first player to obtain a head wins the game. Suppose the first toss is executed by A, what is the probability that A wins the game? Assume independence of the tosses.

7. (2 points) Suppose X and Y take values in $\{0, 1\}$ with joint probability mass function $f(x, y)$. Let $f(0, 0) = a, f(0, 1) = b, f(1, 0) = c$ and $f(1, 1) = d$. Find necessary and sufficient conditions for X and Y to be:
   (a) uncorrelated
   (b) independent

8. (2 points) Let X and Y be independent Poisson variables with parameters λ and μ respectively. Show that
   (a) $X + Y$ is a Poisson random variable with parameter $\lambda + \mu$.
   (b) the conditional distribution of X, given $X + Y = n$ is binomial. Find the parameters of the binomial distribution.

9. (2 points) Let X and Y are two independent Bernoulli random variables with $P(X = 1) = p$ and $P(Y = 1) = q$. Find $E[X|\ X + Y = 1]$.

10. (2 points) If two points are chosen at random from the unit interval $[0, 1]$, what is the probability that the distance between them is less than $\frac{1}{2}$? A random point chosen from the unit interval can be modelled as a uniform random variable on the interval. Assume independence of the two choices.

11. (2 points) Let $X \sim U[-1, 1]$ and $Y \sim U[0, 0.5]$ be independent. What is the probability density function of $X + Y$?

12. (2 points) Let $X_1, X_2, \ldots, X_n$ be independent uniform random variables on $[0, 1]$. Find the expected value of $\max(X_1, X_2, \ldots, X_n)$.

13. (2 points) Consider a pair of random variables X and Y with the following joint probability density function.

$$f_{X,Y}(x, y) = \begin{cases} e^{-y} & 0 \leq x \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

    (a) Find the expectation of X.
    (b) Find the conditional expectation of X given $Y = y$.

14. (2 points) Suppose a biased coin with probability of heads equal to p is tossed repeatedly until there are k consecutive heads. What is the expected number of tosses in this experiment? *Hint: Let* N_k *be the number of tosses until* k *consecutive heads appear. Then* $E[N_k] = E[E[N_k|N_{k-1}]]$

15. (2 points) Give an example of a pair of discrete random variables which are uncorrelated but not independent.