EE 325: Probability and Random Processes (Spring 2014)
Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay

Quiz 3 : 16 points April 11, 2014

---

1. (2 points) Suppose X and Y are independent random variables with characteristic functions

$$\begin{aligned}\phi_X(t) &= \exp\left(i5t - 5t^2\right) \\ \phi_Y(t) &= \exp\left(i6t - 4t^2\right)\end{aligned}$$

respectively. Find the characteristic function of $3X + 4Y + 5$.

2. (4 points) Suppose we have n letters with n matching envelopes i.e. a letter has exactly one matching envelope. A secretary randomly puts the n letters into the envelopes. Let p_n be the probability that none of the letters are placed in their matching envelope.
   (a) Write down a recurrence relation for p_n in terms of p_{n-1} and p_{n-2} for $n \geq 3$. (*Hint: Condition on the first letter going envelope i and the ith letter going to either the first envelope or some other envelope*).
   (b) Let $G(s) = \sum_{n=1}^{\infty} p_n s^n$. Using the above recurrence relation, obtain a differential equation involving $G(s)$.
   (c) Calculate p_1 and p_2 by direct reasoning. Using these values for p_1 and p_2, solve the differential equation for $G(s)$ and get p_n.

3. (2 points) A fair coin is tossed repeatedly and a random process X_n for $n = 1, 2, 3, \dots$ is generated according to the following rule.

$$X_n = \begin{cases} 2^n & \text{if the first } n \text{ tosses all result in heads} \\ 0 & \text{if any one of the first } n \text{ tosses results in tails} \end{cases}$$

   (a) Prove or disprove almost sure convergence of X_n.
   (b) Prove or disprove convergence in probability of X_n.
   (c) Prove or disprove convergence in distribution of X_n.
   (d) Prove or disprove convergence in mean of X_n.

4. (2 points) Consider the following experiment. A fair coin is tossed. If it shows heads, the outcome ω of the experiment is zero. If it shows tails, a uniform random variable from the interval $[-1, 1]$ is generated and the outcome ω is equal to this random variable. Let $X_n(\omega) = 1 + \omega^n$ for $n = 1, 2, 3, \dots$.
   (a) Prove or disprove almost sure convergence of X_n.
   (b) Prove or disprove convergence in probability of X_n.
   (c) Prove or disprove convergence in distribution of X_n.
   (d) Prove or disprove convergence in mean of X_n.

5. (4 points) A fair die with faces $\{1, 2, \dots, 6\}$ is rolled 500 times. Using the central limit theorem, estimate the probability that the sum of the 500 rolls will be at least 1800.

6. (2 points) Let $X(t)$ be a zero-mean wide sense stationary random process with autocorrelation function $R_X(\tau)$. Let $Y(t) = t + X(t)$. Find the mean function and autocorrelation function of $Y(t)$ in terms of $R_X(\tau)$. Prove or disprove the wide sense stationarity of $Y(t)$.