Generating Functions

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Generating Functions

Definition

The generating function of a sequence of real numbers $\{a_i : i = 0, 1, 2, \ldots\}$ is defined by

$$G(s) = \sum_{i=0}^{\infty} a_i s^i$$

for $s \in \mathbb{R}$ for which the sum converges.

Example

Consider the sequence $a_i = 2^{-i}, i = 0, 1, 2, \ldots$.

$$G(s) = \sum_{i=0}^{\infty} \left(\frac{s}{2}\right)^i = \frac{1}{1 - \frac{s}{2}} \qquad \text{for } |s| < 2.$$

Definition

Suppose X is a discrete random variable taking non-negative integer values $\{0, 1, 2, \ldots\}$. The generating function of X is the generating function of its probability mass function.

$$G(s) = \sum_{i=0}^{\infty} P[X = i] s^i = E[s^X]$$

Examples of Generating Functions

- **Constant RV:** Suppose $P(X = c) = 1$ for some fixed $c \in \mathbb{Z}^+$

$$G(s) = E(s^X) = s^c$$

- **Bernoulli RV:** $P(X = 1) = p$ and $P(X = 0) = 1 - p$

$$G(s) = 1 - p + ps$$

- **Geometric RV:** $P(X = k) = p(1 - p)^{k-1}$ for $k \geq 1$

$$G(s) = \sum_{k=1}^{\infty} s^k p(1 - p)^{k-1} = \frac{ps}{1 - s(1 - p)}$$

- **Poisson RV:** $P[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}$ for $k \geq 0$

$$G(s) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!} = e^{\lambda(s-1)}$$

Moments from the Generating Function

Theorem

If X has generating function $G(s)$ then

- $E[X] = G^{(1)}(1)$
- $E[X(X-1)\cdots(X-k+1)] = G^{(k)}(1)$

where $G^{(k)}$ is the kth derivative of $G(s)$.

Result

$\text{var}(X) = G^{(2)}(1) + G^{(1)}(1) - G^{(1)}(1)^2$

Example (Geometric RV)

A geometric RV X has generating function $G(s) = \frac{ps}{1-s(1-p)}$. $\text{var}(X) = ?$

$$\begin{aligned}
G^{(1)}(1) &= \frac{\partial}{\partial s}\frac{ps}{1-s(1-p)}\bigg|_{s=1} = \frac{1}{p} \\
G^{(2)}(1) &= \frac{\partial^2}{\partial s^2}\frac{ps}{1-s(1-p)}\bigg|_{s=1} = \frac{2(1-p)}{p} + \frac{2(1-p)^2}{p^2} \\
\text{var}(X) &= \frac{1-p}{p^2}
\end{aligned}$$

Generating Function of a Sum of Independent RVs

Theorem

If X and Y are independent, $G_{X+Y}(s) = G_X(s)G_Y(s)$

Example (Binomial RV)

Using above theorem, how can we find the generating function of a binomial random variable?
A binomial random variable with parameters n and p is a sum of n independent Bernoulli random variables.

$$S = X_1 + X_2 + \cdots + X_{n-1} + X_n$$

where each X_i has generating function $G(s) = 1 - p + ps = q + ps$.

$$G_S(s) = [G(s)]^n = [q + ps]^n$$

Example (Sum of independent Poisson RVs)

Let X and Y be independent Poisson random variables with parameters λ and μ respectively. What is the distribution of $X + Y$?
Poisson with parameter $\lambda + \mu$

Sum of a Random Number of Independent RVs

Theorem

Let $X_1, X_2, \ldots$ is a sequence of independent identically distributed (iid) random variables with common generating function $G_X(s)$. Let N be a random variable which is independent of the X_i's and has generating function $G_N(s)$. Then

$$S = X_1 + X_2 + \cdots + X_N$$

has generating function given by

$$G_S(s) = G_N\left(G_X(s)\right)$$

Example

A group of hens lay N eggs where N has a Poisson distribution with parameter λ. Each egg results in a healthy chick with probability p independently of the other eggs. Let K be the number of healthy chicks. Find the distribution of K.
Solution Poisson with parameter λp

Joint Generating Function

Definition

The joint generating function of random variables X and Y taking values in the non-negative integers is defined by

$$G_{X,Y}(s_1, s_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} P[X = i, Y = j] s_1^i s_2^j = E[s_1^X s_2^Y]$$

Theorem

Random variables X and Y are independent if and only if

$$G_{X,Y}(s_1, s_2) = G_X(s_1) G_Y(s_2), \quad \text{for all } s_1 \text{ and } s_2.$$

Application: Coin Toss Game

A biased coin which shows heads with probability p is tossed repeatedly. Player A wins if m heads appear before n tails, and player B wins otherwise. What is the probability of A winning?

- Let $p_{m,n}$ be the probability that A wins
- Let $q = 1 - p$. We have the following recurrence relation

$$p_{m,n} = pp_{m-1,n} + qp_{m,n-1}, \quad \text{for } m, n \geq 1$$

- For $m, n > 0$, we have $p_{m,0} = 0$ and $p_{0,n} = 1$. Let $p_{0,0} = 0$.
- Consider the generating function

$$G(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{m,n} x^m y^n$$

- Multiplying the recurrence relation by $x^m y^n$ and sum over $m, n \geq 1$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m,n} x^m y^n = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} pp_{m-1,n} x^m y^n + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} qp_{m,n-1} x^m y^n$$

Coin Toss Game

- Providing the terms corresponding to $m = 0$ and $n = 0$

$$G(x,y) - \sum_{m=1}^{\infty} p_{m,0} x^m - \sum_{n=1}^{\infty} p_{0,n} y^n = px \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m-1,n} x^{m-1} y^n + qy \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{m,n-1} x^m y^{n-1}$$

- Using the boundary conditions we have

$$\begin{aligned} G(x,y) - \frac{y}{1-y} &= pxG(x,y) + qy\left(G(x,y) - \frac{y}{1-y}\right) \\ \Longrightarrow G(x,y) &= \frac{y(1-qy)}{(1-y)(1-px-qy)} \end{aligned}$$

- The coefficient of $x^m y^n$ n $G(x,y)$ gives $p_{m,n}$

Application: Random Walk

- Let $X_1, X_2, \ldots$ be independent random variables taking value 1 with probability p and value -1 with probability $1-p$
- The sequence $S_n = \sum_{i=1}^{n} X_i$ is a random walk starting at the origin
- What is the probability that the walker ever returns to the orgin?
- Let $f_0(n) = \Pr(S_1 \neq 0, \ldots, S_{n-1} \neq 0, S_n = 0)$ be the probability that the first return to the origin occurs after n steps
- Let $p_0(n) = \Pr(S_n = 0)$ be the probability of being at the origin after n steps
- Consider the following generating functions

$$P_0(s) = \sum_{n=0}^{\infty} p_0(n)s^n, \qquad F_0(s) = \sum_{n=0}^{\infty} f_0(n)s^n.$$

- $P_0(s) = 1 + P_0(s)F_0(s)$
- $P_0(s) = (1 - 4pqs^2)^{-\frac{1}{2}}$
- $F_0(s) = 1 - (1 - 4pqs^2)^{\frac{1}{2}}$
- $\sum_{n=1}^{\infty} f_0(n) = F_0(1) = 1 - |p - q|$

Reference

- Chapter 5, *Probability and Random Processes*, Grimmett and Stirzaker, Third Edition, 2001.