Power Spectral Density

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

April 4, 2014

Power Spectral Density

- Fourier transform

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft)\, dt \\
X(f) &= \mathcal{F}(x(t))
\end{aligned}
$$

- Inverse Fourier transform

$$
\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft)\, df \\
x(t) &= \mathcal{F}^{-1}(X(f))
\end{aligned}
$$

Definition (Power Spectral Density of a WSS Process)

The power spectral density of a wide-sense stationary random process is the Fourier transform of the autocorrelation function.

$$S_X(f) = \mathcal{F}(R_X(\tau))$$

Motivating the Definition of Power Spectral Density

$$X(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(t)$$

- Consider an LTI system with impulse response $h(t)$ which has random processes $X(t)$ and $Y(t)$ as input and output

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)\, d\tau$$

- If $X(t)$ is a wide-sense stationary random process, then $Y(t)$ is also wide-sense stationary with autocorrelation function

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)\, d\tau_1\, d\tau_2$$

- Setting $\tau = 0$, we can express the average power in the output process as

$$R_Y(0) = E\left[Y^2(t)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\, d\tau_1\, d\tau_2$$

Motivating the Definition of Power Spectral Density

- Let $H(f)$ be the Fourier transform of the impulse response $h(t)$

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi f \tau_1)\, df$$

- Substituting the above equation into the average power equation we get

$$\begin{aligned}
E\left[Y^2(t)\right] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau_2-\tau_1)\, d\tau_1\, d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(f)e^{j2\pi f\tau_1}\, df\right] h(\tau_2)R_X(\tau_2-\tau_1)\, d\tau_1\, d\tau_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f)\left[\int_{-\infty}^{\infty} h(\tau_2)e^{j2\pi f\tau_2}\, d\tau_2\right] R_X(\tau)e^{-j2\pi f\tau}\, d\tau\, df \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(f)H^*(f)R_X(\tau)e^{-j2\pi f\tau}\, d\tau\, df \\
&= \int_{-\infty}^{\infty} |H(f)|^2 \int_{-\infty}^{\infty} R_X(\tau)e^{-j2\pi f\tau}\, d\tau\, df \\
&= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df
\end{aligned}$$

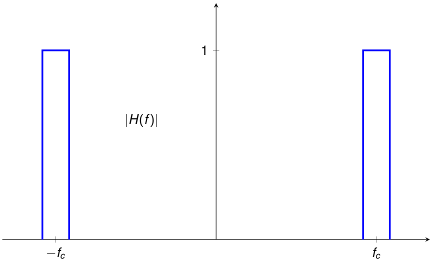
Motivating the Definition of Power Spectral Density

- The output power and power spectral density are related by

$$E\left[Y^2(t)\right] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f)\, df$$

- Let the LTI system be an ideal narrowband filter with magnitude response given by

$$|H(f)| = \begin{cases} 1 & |f \pm f_c| \leq \frac{\Delta f}{2} \\ 0 & |f \pm f_c| > \frac{\Delta f}{2} \end{cases}$$



$$E\left[Y^2(t)\right] \approx (2\Delta f) S_X(f_c)$$

Properties of Power Spectral Density

- The power spectral density and autocorrelation function form a Fourier transform pair

$$
\begin{aligned}
S_X(f) &= \int_{-\infty}^{\infty} R_X(\tau) \exp(-i2\pi f\tau)\, d\tau \\
R_X(\tau) &= \int_{-\infty}^{\infty} S_X(f) \exp(i2\pi f\tau)\, df
\end{aligned}
$$

- Power spectral density is a non-negative and even function of f
- Zero-frequency PSD value equals area under autocorrelation function

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

- Power of $X(t)$ equals area under power spectral density

$$E\left[X^2(t)\right] = \int_{-\infty}^{\infty} S_X(f)\, df$$

- If $X(t)$ is passed through an LTI system with frequency response $H(f)$ to get $Y(t)$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

White Noise

- A wide-sense stationary random process with flat power spectral density
$$S_W(f) = \frac{N_0}{2}$$
where N_0 has units Watts per Hertz
- White noise has infinite power and is not physically realizable
- Models a situation where the noise bandwidth is much larger than the signal bandwidth
- The corresponding autocorrelation function is given by
$$R_N(\tau) = \frac{N_0}{2}\delta(\tau)$$
where δ is the Dirac delta function

Reference

- Chapter 1, *Communication Systems*, Simon Haykin, Fourth Edition, Wiley-India, 2001.