# Properties of Probability Measures

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

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# Probability Space

## Definition

A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of a set $\Omega$, a $\sigma$-field $\mathcal{F}$ of subsets of $\Omega$ and a probability measure $P$ on $(\Omega, \mathcal{F})$.

## Definition

A probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ satisfying

(a) $P(\phi) = 0$, $P(\Omega) = 1$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

# Some Properties of Probability Measures

- $P(A^c) = 1 - P(A)$
- Define $B \setminus A = B \cap A^c$. If $A \subseteq B$, then $P(B) = P(A) + P(B \setminus A) \geq P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- For any $n \in \mathbb{N}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_i P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots A_n)$$

# $P$ is a continuous set function

## Theorem

*Let $A_1, A_2, \ldots$ be an increasing sequence of events, so that $A_1 \subseteq A_2 \subseteq \cdots$. Let $A$ be their limit*

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i.$$

*Then $P(A) = \lim_{i \to \infty} P(A_i)$.*

## Proof.

The set $A$ can be written as a disjoint union as follows

$$A = A_1 \bigcup (A_2 \setminus A_1) \bigcup (A_3 \setminus A_2) \bigcup (A_4 \setminus A_3) \cdots$$

By the countable additivity property of $P$, we have

$$\begin{aligned} P(A) &= P(A_1) + \sum_{i=1}^{\infty} P(A_{i+1} \setminus A_i) = P(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n} P(A_{i+1} \setminus A_i) \\ &= P(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n} [P(A_{i+1}) - P(A_i)] = P(A_1) + \lim_{n \to \infty} [P(A_{n+1}) - P(A_1)] \\ &= \lim_{n \to \infty} P(A_{n+1}) \end{aligned}$$

# $P$ is a continuous set function

## Theorem

*Let $B_1, B_2, \ldots$ be a decreasing sequence of events, so that $B_1 \supseteq B_2 \supseteq \cdots$.*
*Let $B$ be their limit*

$$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i.$$

*Then $P(B) = \lim_{i \to \infty} P(B_i)$.*

## Proof.

Let $A_i = B_i^c$ and use the previous theorem.

Questions?