EE 325: Probability and Random Processes (Spring 2015)
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Assignment 4: 25 points — Due Date: March 30, 2015

1. Suppose the input X and output Y to a channel are related by $Y = \rho X + N$ where N is a zero-mean Gaussian random variable with variance σ^2 and ρ is a random variable independent of the noise. Assume that X is equally likely to be $\pm A$. Our goal is to decide on the value of X given the observation Y.
   (a) [2½ points] If ρ is the constant 1, what is the optimal decision rule and the resulting decision error probability?
   (b) [2½ points] If ρ takes values ± 1 with equal probability, what is the optimal decision rule and the resulting decision error probability?

2. [5 points] Find the maximum likelihood decision rule for the following 3-ary hypothesis testing problem where $\mu = \sqrt{2}\sigma$.

$$\begin{aligned} H_1 &: Y \sim N(-\mu, \sigma^2) \\ H_2 &: Y \sim N(0, e^2\sigma^2) \\ H_3 &: Y \sim N(\mu, \sigma^2) \end{aligned}$$

   Hint: Sketch the density functions keeping in mind that the variances are unequal.

3. Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$\begin{aligned} H_0 &: Y \sim U\left[-\sqrt{\frac{e^2\pi}{2}}, \sqrt{\frac{e^2\pi}{2}}\right] \\ H_1 &: Y \sim \mathcal{N}(0, 1) \end{aligned}$$

   U denotes the uniform distribution, $\mathcal{N}$ denotes the Gaussian distribution and e is the base of the natural logarithm.
   (a) [2½ points] Find the decision error probability of the rule which decides H_1 is true if $|Y| > \sqrt{\frac{e^2\pi}{2}}$ and decides H_0 is true if $|Y| \leq \sqrt{\frac{e^2\pi}{2}}$. Express your answer in terms of the Q function.
   (b) [2½ points] Find the decision error probability of the optimal decision rule. Express your answer in terms of the Q function.

4. [5 points] Suppose observations Y_i, $i = 1, 2, \ldots, N$ are Poisson distributed with parameter λ. Assume that the Y_i's are independent.
   (a) Derive the ML estimator for λ.
   (b) Find the mean and variance of the ML estimate.

5. [5 points] Suppose we observe a sequence of random variables $Y_1, Y_2, \ldots, Y_n$ given by

$$Y_k = \theta s_k + N_k, \quad k = 1, 2, \ldots, n$$

   where the N_k's are independent zero-mean Gaussian random variables with variance σ^2. The sequence $s_1, \ldots, s_n$ is a known signal sequence and θ is an unknown parameter.
   (a) Find the maximum likelihood estimate $\hat{\theta}_{ML}(\mathbf{Y})$ of the parameter θ.
   (b) Find the mean and variance of $\hat{\theta}_{ML}(\mathbf{Y})$.