EE 325: Probability and Random Processes (Spring 2015)
Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay

Endsemester Exam : 40 points April 25, 2015

---

1. (4 points) A fair coin is tossed $4N$ times. Assume that the tosses are independent. Using Markov and Chebyshev's inequalities (separately), calculate upper bounds on the probability of the event that at least $3N$ heads occur in the $4N$ tosses. Which upper bound is tighter?

2. (4 points) Prove that if X is a Gaussian random variable, then $aX + b$ is also a Gaussian random variable for $a, b \in \mathbb{R}, a \neq 0$. Your proof should hold even when $a < 0$.

3. (4 points) Consider the following binary hypothesis testing problem where the hypotheses are equally likely and $\lambda_1 > \lambda_2$.

$$\begin{aligned} H_1 &: X_i \sim \text{Poisson}(\lambda_1), \quad i = 1, 2, \ldots, N \\ H_2 &: X_i \sim \text{Poisson}(\lambda_2), \quad i = 1, 2, \ldots, N \end{aligned}$$

Assume that X_i and X_j are independent for $i \neq j$. The probability mass function of a Poisson random variable Z with parameter γ is given by $P(Z = n) = \frac{\gamma^n}{n!} e^{-\gamma}$ for $n = 0, 1, 2, 3, \ldots$.

   (a) Find the optimal decision rule which minimizes the decision error probability. Simplify it as much as possible.

   (b) Let $F(x; \gamma)$ be the cumulative distribution function of a Poisson random variable with parameter γ. Find the decision error probability of the optimal decision rule in terms of F, λ_1, λ_2, and N. *Hint: Sum of two independent Poisson random variables with parameters γ_1 and γ_2 is a Poisson random variable with parameter $\gamma_1 + \gamma_2$.*

4. (4 points) Suppose we observe $Y_i, \ i = 1, 2, \ldots, M$ such that

$$Y_i \sim U[\theta_1, \theta_2]$$

where the Y_i's are independent and θ_1, θ_2 are unknowns. Derive the maximum-likelihood estimators of θ_1 and θ_2. *Hint: Imagine a 3D plot with θ_2 on the x-axis, θ_1 on the y-axis and the likelihood $p_{\theta_1,\theta_2}(y_i)$ on the z-axis. First find the region in the $\theta_1\theta_2$ plane where the joint likelihood $p_{\theta_1,\theta_2}(y_1, y_2, \ldots, y_M)$ is non-zero and then identify where it reaches its maximum.*

5. (4 points) A random variable S is generated by the following procedure:

   - Generate two independent uniform random variables V_1 and V_2 in the interval $[-1, 1]$.
   - If $V_1^2 + V_2^2 > 1$, then go to step 1 and regenerate V_1 and V_2.
   - If $V_1^2 + V_2^2 \leq 1$, then set $S = V_1^2 + V_2^2$ and stop.

   Show that S has a uniform distribution in the interval $[0, 1]$. *Hint: The above procedure was used in the Box-Muller method for generating Gaussian random variables.*

6. (4 points) In quiz 3, you saw that $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$ **does not imply** $X_n + Y_n \xrightarrow{D} X + Y$. Show that if $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} c$ where $c \in \mathbb{R}$ is a constant, then $X_n + Y_n \xrightarrow{D} X + c$. You can use any result stated in quiz 3 without proof. *Hint: Condition on the partition $|Y_n - c| \leq \varepsilon$ and $|Y_n - c| > \varepsilon$.*

7. (4 points) Let X_n for $n \in \mathbb{N}$ be a sequence of independent but **not identically distributed** random variables having distribution

$$X_n \sim \begin{cases} U[0, 2] & \text{if } n \text{ is odd,} \\ U[2, 4] & \text{if } n \text{ is even} \end{cases}$$

where $U[a, b]$ represents the uniform distribution in the interval $[a, b]$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Show that $\frac{S_n}{n}$ converges in probability to a constant. You have to specify the constant explicitly. You can use any result stated in quiz 3 without proof.

8. (4 points) Let X_n for $n \in \mathbb{Z}$ be a random process consisting of independent and identically distributed random variables. Show that X_n is a strict-sense stationary random process.

9. (4 points) Let X_n for $n \in \mathbb{Z}$ be a wide-sense stationary random process with zero mean function and autocorrelation function given by

$$R_X[k] = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_n = \frac{X_{n-1} + X_n + X_{n+1}}{3}$ be a filtered version of X_n. Calculate the autocorrelation function of Y_n.

10. (4 points) Two random vectors $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_m \end{bmatrix}$ are independent if their joint probability density or mass function is a product of the marginal density or mass functions i.e. $p(\mathbf{X}, \mathbf{Y}) = p(\mathbf{X})p(\mathbf{Y})$.

Two random processes $X(t)$ and $Y(t)$ are independent if any two vectors of time samples are independent i.e. $\begin{bmatrix} X(t_1) & X(t_2) & \cdots & X(t_n) \end{bmatrix}$ and $\begin{bmatrix} Y(\tau_1) & Y(\tau_2) & \cdots & Y(\tau_m) \end{bmatrix}$ are independent vectors as per the previous definition for any $n, m \in \mathbb{N}$ and any $t_1, t_2, \ldots, t_n, \tau_1, \tau_2, \ldots, \tau_m \in \mathbb{R}$.

Suppose $X(t)$ and $Y(t)$ are independent wide-sense stationary random processes with mean functions equal to μ_X and μ_Y respectively. Let their autocorrelation functions be $R_X(\tau)$ and $R_Y(\tau)$ respectively.

(a) Show that $Z(t) = X(t) + Y(t)$ is a wide-sense stationary random process.
which depends only on τ. Hence $Z(t)$ is a wide-sense stationary random process.

(b) Show that $W(t) = X(t)Y(t)$ is a wide-sense stationary random process.