EE 325: Probability and Random Processes (Spring 2015)
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Quiz 2 : 30 points April 1, 2015

1. (6 points) The coefficient of correlation $\rho(X,Y)$ between random variables X and Y is defined as
$$\rho(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sigma(X)\sigma(Y)}$$
where $\sigma(X)$ and $\sigma(Y)$ are the standard deviations of X and Y respectively. Show that $\rho(X,Y)$ always lies in the interval $[-1,1]$.

2. (6 points) Consider the following ternary hypothesis testing problem where the hypotheses are equally likely.
$$\begin{aligned} H_1 &: Y \sim \mathcal{N}(-\mu, \sigma^2) \\ H_2 &: Y \sim \mathcal{N}(0, \sigma^2) \\ H_3 &: Y \sim \mathcal{N}(\mu, \sigma^2) \end{aligned}$$
Here Y is the observation and $\mathcal{N}(\mu,\sigma^2)$ denotes the Gausssian distribution with mean μ and variance σ^2.
   (a) Find the optimal decision rule which minimizes the decision error probability. Simplify it as much as possible.
   (b) Find the decision error probability of the optimal decision rule in terms of the Q function.

3. (6 points) Consider the following binary hypothesis testing problem where hypothesis H_0 is true with probability $\frac{1}{3}$ and hypothesis H_1 is true with probability $\frac{2}{3}$.
$$\begin{aligned} H_0 &: Y \sim U[0,1] \\ H_1 &: Y \sim U[0.5, 2.5] \end{aligned}$$
Here Y is the observation and $U[a,b]$ denotes the uniform distribution in the interval $[a,b]$.
   (a) Find the optimal decision rule which minimizes the decision error probability. Simplify it as much as possible.
   (b) Find the decision error probability of the optimal decision rule.

4. (6 points) Suppose observations Y_i, $i = 1, 2, \ldots, N$ are Rayleigh distributed with parameter σ^2. A Rayleigh distributed random variable with parameter $\sigma^2 > 0$ has probability density function given by
$$p(y) = \frac{y}{\sigma^2} e^{-y^2/2\sigma^2} \text{ for } y \geq 0.$$
The mean and variance of a Rayleigh distributed random variable are given by $\sigma\sqrt{\frac{\pi}{2}}$ and $\frac{4-\pi}{2}\sigma^2$ respectively. Assume that the Y_i's are independent.
   (a) Derive the maximum likelihood estimator for σ^2.
   (b) Show that the expected value of the maximum likelihood estimate is σ^2.

5. (6 points) Suppose observations X_i and Y_i ($i = 1, \ldots, N$) are Poisson distributed as follows where $\lambda > 0$ is an unknown parameter.
$$\begin{aligned} X_i &\sim \text{Poisson}(\lambda), \quad i = 1, 2, \ldots, N \\ Y_i &\sim \text{Poisson}(2\lambda), \quad i = 1, 2, \ldots, N \end{aligned}$$
The probability mass function of a Poisson random variable Z with parameter γ i.e. $Z \sim \text{Poisson}(\gamma)$ is given by $P(Z = n) = \frac{\gamma^n}{n!}e^{-\gamma}$ for $n = 0, 1, 2, 3, \ldots$. Assume that X_i and X_j are independent for $i \neq j$. Assume that Y_i and Y_j are independent for $i \neq j$. Assume that X_i and Y_j are independent for all i, j.
   (a) Derive the maximum likelihood estimator for λ.
   (b) Find the mean and variance of the maximum likelihood estimate.