EE 325: Probability and Random Processes (Spring 2015)
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Quiz 3 : 30 points April 17, 2015

1. (6 points) Consider a biased coin which shows heads with probability p when tossed. Suppose the coin is tossed until a heads appears. Let X be the number of tosses in the experiment (including the last toss where the heads appears). Given p, describe a procedure to generate the random variable X. You can assume that the tosses are independent.

2. (6 points) Let X_n for $n = 1, 2, 3, \dots$, be a sequence of random variables. Show that $X_n \xrightarrow{D} c$ implies $X_n \xrightarrow{P} c$ where c is a constant.

3. (6 points) State and prove the weak law of large numbers. You can use the result in question 2.

4. (6 points) State and prove the central limit theorem.

5. (6 points) Let X_n and Y_n for $n = 1, 2, 3, \dots$, be sequences of random variables.

   (a) Suppose $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$. Prove that $X_n + Y_n \xrightarrow{P} X + Y$.
   In terms of the sample points, the above statement is

$$\left\{\omega \in \Omega \middle| |X_n(\omega) + Y_n(\omega) - X(\omega) - Y(\omega)| > \varepsilon\right\} \subseteq \left\{\omega \in \Omega \middle| |X_n(\omega) - X(\omega)| > \frac{\varepsilon}{2}\right\} \bigcup \left\{\omega \in \Omega \middle| |Y_n(\omega) - Y(\omega)| > \frac{\varepsilon}{2}\right\}.$$

   Suppose this statement is not true. Then there exists an sample point $\omega_0 \in \Omega$ such that it belongs to the set on the left hand side of the set inclusion but not to the union on the right hand side. If the ω_0 does not belong to the union on the right hand side, then $|X_n(\omega_0) - X(\omega_0)| \le \frac{\varepsilon}{2}$ and $|Y_n(\omega_0) - Y(\omega_0)| \le \frac{\varepsilon}{2}$. This implies

$$|X_n(\omega_0) - X(\omega_0)| + |Y_n(\omega_0) - Y(\omega_0)| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

   This is a contradiction since ω_0 belongs to the left hand side which implies

$$\varepsilon < |X_n(\omega_0) + Y_n(\omega_0) - X(\omega_0) - Y(\omega_0)| \le |X_n(\omega_0) - X(\omega_0)| + |Y_n(\omega_0) - Y(\omega_0)| \le \varepsilon.$$

   Now that we have proved that $\{|X_n + Y_n - X - Y| > \varepsilon\} \subseteq \{|X_n - X| > \frac{\varepsilon}{2}\} \bigcup \{|Y_n - Y| > \frac{\varepsilon}{2}\}$, we can apply the union bound to get

$$\Pr\{|X_n + Y_n - X - Y| > \varepsilon\} \le \Pr\left\{|X_n - X| > \frac{\varepsilon}{2}\right\} + \Pr\left\{|Y_n - Y| > \frac{\varepsilon}{2}\right\}.$$

   Since $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, the right hand side in the union bound goes to zero as $n \to \infty$. This proves that $X_n + Y_n \xrightarrow{P} X + Y$.

   (b) Give a counterexample to show that $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$ **does not imply** $X_n + Y_n \xrightarrow{D} X + Y$.