Combinatorial Probability

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

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Equally Likely Outcomes in a Finite Sample Space

- Many interesting experiments have a finite sample space and equally likely outcomes
- The σ-field $\mathcal{F}$ in this case is the power set of the sample space
- The probability of an event is the ratio of the cardinalities of the event and the sample space

  $$P(A) = \frac{|A|}{|\Omega|}$$

  for $A \subseteq \Omega$
- In this situation, to find the probability of an event A we need to find its cardinality i.e. count the number of elements in it

Fundamental Rule

- A number of multiple choices are to be made.
- There are m_1 possibilities for the first choice, m_2 for the second, m_3 for the third and so on.
- If these choices can be combined freely, then the total number of possibilities for the whole set of choices is equal to

$$m_1 \times m_2 \times m_3 \times \cdots .$$

Examples

- In how many ways can five dice appear when they are rolled?
- In how many ways can five dice show different faces when they are rolled?
- A man has three shirts and two ties. How many ways can he dress up?

Sampling Models

- An urn contains m distinguishable balls numbered 1 to m
- n balls will be drawn under various specified conditions
  - Sampling with replacement and with ordering
  - Sampling without replacement and with ordering
  - Sampling without replacement and without ordering
  - Sampling with replacement and without ordering
- Number of all possible outcomes will be counted in each case

Sampling with Replacement and with Ordering

- The n balls are drawn sequentially, each drawn ball being put back into the urn before the next drawing
- The numbers on the balls are recorded in the order of their appearance
- One outcome of this experiment can be described by an n-tuple $(a_1, a_2, \ldots, a_n)$ where each a_i can be any integer from 1 to m
- By the fundamental rule, the total number of outcomes is

$$\underbrace{m \times m \times \cdots \times m}_{n \text{ times}} = m^n$$

Sampling without Replacement and with Ordering

- The n balls are drawn sequentially, each drawn ball is left out of the urn for the next drawing
- The numbers on the balls are recorded in the order of their appearance
- One outcome of this experiment can be described by an n-tuple $(a_1, a_2, \ldots, a_n)$ where the a_i's are all different integers between 1 and m
- By the fundamental rule, the total number of outcomes is

$$m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)$$

- If $n = m$, such a sampling is called a permutation. The total number of permutations is

$$m! = m \cdot (m-1) \cdot (m-2) \cdots 2 \cdot 1$$

Sampling without Replacement and without Ordering

- The n balls are drawn sequentially, each drawn ball is left out of the urn for the next drawing
- The order of appearance of the balls is not recorded. This is equivalent to grabbing n balls in one go.
- One outcome of this experiment can be described by a set $\{a_1, a_2, \ldots, a_n\}$ where the a_i's are all different integers between 1 and m
- Let x be the total number of outcomes
- By permuting each group of n balls, we can get any n-tuple which contains these n balls
- Thus $x \cdot n! = m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)$
- The total number of outcomes is

$$x = \frac{m \cdot (m-1) \cdot (m-2) \cdots (m-n+1)}{n!} = \frac{m!}{n!(m-n)!} = \binom{m}{n}$$

- The number $\binom{m}{n}$ is called a binomial coefficient

Sampling with Replacement and without Ordering

- The n balls are drawn sequentially, each drawn ball being put back into the urn before the next drawing
- The order of appearance of the balls is not recorded
- One outcome of this experiment can be described by an m-tuple $(a_1, a_2, \ldots, a_m)$ where each a_i can be any integer from 0 to n and $\sum_{i=1}^{m} a_i = n$
- The total number of outcomes is

$$\binom{m+n-1}{n}$$

Summary of Sampling Models

Choosing n out of m

	Without Replacement	With Replacement
Ordered	$\frac{m!}{(m-n)!}$	m^n
Unordered	$\binom{m}{n}$	$\binom{m+n-1}{n}$

Examples

- A deck of 52 cards is shuffled. What is the probability that the four aces are found in a row?
- Six dice are rolled. What is the probability of getting three pairs?
- What is the probability that among n people there are at least two who have the same birthday? Ignore leap years.
- A biased coin has a probability p of showing heads when tossed. In n independent tosses of the coin, what is the probability that the kth heads appears on the nth toss?
- A fair coin is tossed n times each by two people. What is the probability that they get the same number of heads?
- One hundred fish are caught from a pond and returned to the pond after they are tagged. Later another 100 fish are caught from the same pond and are found to contain 10 tagged ones. What is the probability of this event if the pond has n fish?

Reading Assignment

- Chapter 3, *Elementary Probability Theory*, K. L. Chung and F. AitSahlia, 2002 (4th Edition)

Questions?