Gaussian Random Variables

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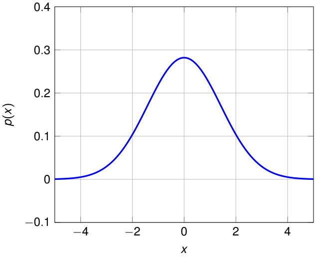
Gaussian Random Variable

Definition

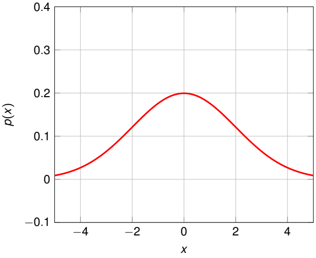
A continuous random variable with probability density function of the form

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

where μ is the mean and σ^2 is the variance.



: $\mu = 0, \sigma^2 = 2$



: $\mu = 0, \sigma^2 = 4$

Notation

- $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2
- $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow X$ is a Gaussian RV with mean μ and variance σ^2
- If $X \sim \mathcal{N}(0, 1)$, then X is a standard Gaussian RV

Affine Transformations Preserve Gaussianity

Theorem

If X is Gaussian, then $aX + b$ is Gaussian for $a, b \in \mathbb{R}, a \neq 0$.

Remarks

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$.

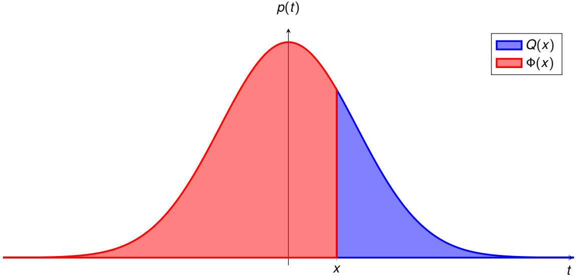
CDF and CCDF of Standard Gaussian

- Cumulative distribution function of $X \sim \mathcal{N}(0,1)$

$$\Phi(x) = P[X \le x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

- Complementary cumulative distribution function of $X \sim \mathcal{N}(0,1)$

$$Q(x) = P[X > x] = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$



Properties of $Q(x)$

- $\Phi(x) + Q(x) = 1$
- $Q(-x) = \Phi(x) = 1 - Q(x)$
- $Q(0) = \frac{1}{2}$
- $Q(\infty) = 0$
- $Q(-\infty) = 1$
- $X \sim \mathcal{N}(\mu, \sigma^2)$

$$P[X > \alpha] = Q\left(\frac{\alpha - \mu}{\sigma}\right)$$

$$P[X < \alpha] = Q\left(\frac{\mu - \alpha}{\sigma}\right)$$

Jointly Gaussian Random Variables

Definition (Jointly Gaussian RVs)

Random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if any non-trivial linear combination is a Gaussian random variable.

$$a_1 X_1 + \cdots + a_n X_n \text{ is Gaussian for all } (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus \mathbf{0}$$

Example (Not Jointly Gaussian)

$X \sim \mathcal{N}(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim \mathcal{N}(0, 1)$ and $X + Y$ is not Gaussian.

Remarks

- Independent Gaussian random variables are always jointly Gaussian
- Knowledge of mean and variance of a linear combination of jointly Gaussian random variables is sufficient to determine it density

Gaussian Random Vector

Definition (Gaussian Random Vector)

A random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ whose components are jointly Gaussian.

Notation

$\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = E[\mathbf{X}], \quad \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T\right]$$

$\mathbf{m}$ is called the mean vector and $\mathbf{C}$ is called the covariance matrix
The joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x} - \mathbf{m})\right)$$

Example (Bivariate Standard Normal Distribution)

X and Y are jointly Gaussian random variables. $[X\ Y]^T \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

What is the joint density? What are the marginal densities of X and Y?

Uncorrelated Jointly Gaussian RVs are Independent

If $X_1, \ldots, X_n$ are jointly Gaussian and pairwise uncorrelated, then they are independent.

$$
\begin{aligned}
p(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^m \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i - m_i)^2}{2\sigma_i^2}\right)
\end{aligned}
$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \mathrm{var}(X_i)$.

Uncorrelated Gaussian RVs may not be Independent

Example

- $X \sim \mathcal{N}(0, 1)$
- W is equally likely to be +1 or -1
- W is independent of X
- $Y = WX$
- $Y \sim \mathcal{N}(0, 1)$
- X and Y are uncorrelated
- X and Y are not independent

Thanks for your attention