Random Processes

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Random Process

Definition

An indexed collection of random variables $\{X_t : t \in \mathcal{T}\}$.

Discrete-time Random Process

A random process where the index set $\mathcal{T} = \mathbb{Z}$ or $\{0, 1, 2, 3, \ldots\}$.

Example: Random walk
$\mathcal{T} = \{0, 1, 2, 3, \ldots\}$, $X_0 = 0$, X_n independent and equally likely to be ± 1 for $n \geq 1$

$$S_n = \sum_{i=0}^{n} X_i$$

Continuous-time Random Process

A random process where the index set $\mathcal{T} = \mathbb{R}$ or $[0, \infty)$. The notation $X(t)$ is used to represent continuous-time random processes.

Example: Thermal Noise

Realization of a Random Process

- The outcome of an experiment is specified by a sample point ω in the sample space Ω
- A realization of a random variable X is its value $X(\omega)$
- A realization of a random process X_t is the function $X_t(\omega)$ of t
- A realization is also called a sample function of the random process.

Example

Consider $\Omega = [0, 1]$. For each $\omega \in \Omega$, consider its dyadic expansion

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n} = 0.d_1(\omega)d_2(\omega)d_3(\omega)\cdots$$

where each $d_n(\omega)$ is either 0 or 1.
An infinite number of coin tosses with Heads being 0 and Tails being 1 can be associated with each ω as

$$X_n(\omega) = d_n(\omega)$$

For each $\omega \in \Omega$, we get a realization of this random process.

Specification of a Random Process

- A random process is specified by the joint cumulative distribution of the random variables

$$X(t_1), X(t_2), \ldots, X(t_n)$$

  for any set of sample times $\{t_1, t_2, \ldots, t_n\}$ and any $n \in \mathbb{N}$

$$F_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n) = \Pr\left[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n\right]$$

- For continuous-time random processes, the joint probability density is sufficient
- For discrete-time random processes, the joint probability mass function is sufficient
- Without additional restrictions, this requires specifying a lot of joint distributions
- One restriction which simplifies process specification is stationarity

Stationary Random Process

Definition

A random process $X(t)$ is said to be *stationary in the strict sense* or *strictly stationary* if the joint distribution of $X(t_1), X(t_2), \ldots, X(t_k)$ is the same as the joint distribution of $X(t_1+\tau), X(t_2+\tau), \ldots, X(t_k+\tau)$ for all time shifts τ, all k, and all observation instants $t_1, \ldots, t_k$.

$$F_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1,\ldots,x_k)$$

Properties

- A stationary random process is statistically indistinguishable from a delayed version of itself.
- For $k = 1$, we have
  $$F_{X(t)}(x) = F_{X(t+\tau)}(x)$$
  for all t and τ. The first order distribution is independent of time.
- For $k = 2$ and $\tau = -t_1$, we have
  $$F_{X(t_1),X(t_2)}(x_1,x_2) = F_{X(0),X(t_2-t_1)}(x_1,x_2)$$
  for all t_1 and t_2. The second order distribution depends only on $t_2 - t_1$.

Mean Function

- The mean of a random process $X(t)$ is the expectation of the random variable obtained by observing the process at time t

$$\mu_X(t) = E\left[X(t)\right] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx$$

- For a strictly stationary random process $X(t)$, the mean is a constant

$$\mu_X(t) = \mu \quad \text{for all } t$$

Example

$X(t) = \cos\left(2\pi f t + \Theta\right)$, $\Theta \sim U[-\pi, \pi]$. $\mu_X(t) = ?$

Example

$X_n = Z_1 + \cdots + Z_n$, $n = 1, 2, \ldots$
where Z_i are i.i.d. with zero mean and variance σ^2. $\mu_X(n) = ?$

Autocorrelation Function

- The autocorrelation function of a random process $X(t)$ is defined as

$$R_X(t_1, t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1\, dx_2$$

- For a strictly stationary random process $X(t)$, the autocorrelation function depends only on the time difference $t_2 - t_1$

$$R_X(t_1, t_2) = R_X(0, t_2 - t_1) \quad \text{for all } t_1, t_2$$

  In this case, $R_X(0, t_2 - t_1)$ is simply written as $R_X(t_2 - t_1)$

Example

$X(t) = \cos(2\pi f t + \Theta)$, $\Theta \sim U[-\pi, \pi]$. $R_X(t_1, t_2) = ?$

Example

$X_n = Z_1 + \cdots + Z_n$, $n = 1, 2, \ldots$
where Z_i are i.i.d. with zero mean and variance σ^2. $R_X(n_1, n_2) = ?$

Wide-Sense Stationary Random Process

Definition

A random process $X(t)$ is said to be *wide-sense stationary* or *weakly stationary* or *second-order stationary* if

$$\begin{aligned} \mu_X(t) &= \mu_X(0) && \text{for all } t \text{ and} \\ R_X(t_1, t_2) &= R_X(t_1 - t_2, 0) && \text{for all } t_1, t_2. \end{aligned}$$

Remarks

- A strictly stationary random process is also wide-sense stationary if the first and second order moments exist.
- A wide-sense stationary random process need not be strictly stationary.

Example

Is the following random process wide-sense stationary?

$$X(t) = A\cos\left(2\pi f_c t + \Theta\right)$$

where A and f_c are constants and Θ is uniformly distributed on $[-\pi, \pi]$.

Properties of the Autocorrelation Function

- Consider the autocorrelation function of a wide-sense stationary random process $X(t)$

$$R_X(\tau) = E\left[X(t+\tau)X(t)\right]$$

- $R_X(\tau)$ is an even function of τ

$$R_X(\tau) = R_X(-\tau)$$

- $R_X(\tau)$ has maximum magnitude at $\tau = 0$

$$|R_X(\tau)| \leq R_X(0)$$

- The autocorrelation function measures the interdependence of two random variables obtained by measuring $X(t)$ at times τ apart

Ergodic Processes

- Let $X(t)$ be a wide-sense stationary random process with mean μ_X and autocorrelation function $R_X(\tau)$ (also called the ensemble averages)
- Let $x(t)$ be a realization of $X(t)$
- For an observation interval $[-T, T]$, the time average of $x(t)$ is given by

$$\mu_x(T) = \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

- The process $X(t)$ is said to be ergodic in the mean if $\mu_x(T)$ converges to μ_X in the squared mean as $T \to \infty$
- For an observation interval $[-T, T]$, the time-averaged autocorrelation function is given by

$$R_x(\tau, T) = \frac{1}{2T} \int_{-T}^{T} x(t+\tau)x(t)\, dt$$

- The process $X(t)$ is said to be ergodic in the autocorrelation function if $R_x(\tau, T)$ converges to $R_X(\tau)$ in the squared mean as $T \to \infty$

Passing a Random Process through an LTI System

$$X(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow Y(t)$$

- Consider a linear time-invariant (LTI) system $h(t)$ which has random processes $X(t)$ and $Y(t)$ as input and output

$$Y(t) = \int_{-\infty}^{\infty} h(\tau) X(t-\tau)\, d\tau$$

- In general, it is difficult to characterize $Y(t)$ in terms of $X(t)$
- If $X(t)$ is a wide-sense stationary random process, then $Y(t)$ is also wide-sense stationary

$$\begin{aligned} \mu_Y(t) &= \mu_X \int_{-\infty}^{\infty} h(\tau)\, d\tau \\ R_Y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 + \tau_2)\, d\tau_1\, d\tau_2 \end{aligned}$$

Reference

- Chapter 1, *Communication Systems*, Simon Haykin, Fourth Edition, Wiley-India, 2001.