EE 325: Probability and Random Processes (Spring 2025)
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Assignment 1: 20 points — Date: January 14, 2025

1. [5 points] Consider the set B of all binary sequences given by

$$B = \{b_1 b_2 b_3 b_4 \cdots \mid b_i \in \{0,1\} \text{ for } i \in \mathbb{N}\}.$$

   Show that B is uncountable. *Hint: Use Cantor's diagnolization argument.*

2. [5 points] Let $A_1, A_2, A_3, A_4, \ldots$ be a sequence of countable sets. Show that $\bigcup_{i=1}^{\infty} A_i$ is countable.

   Hint: Recall the proof of the countability of $\mathbb{N} \times \mathbb{N}$. *There* $\{(i,1),(i,2),(i,3),\ldots\}$ *was a countable set of each* $i \in \{1,2,3,\ldots\}$.

3. [5 points] Consider the equivalence relation R on the interval $[0,1]$ given by $x \sim y$ if $x - y$ is rational. This equivalence relation partitions $[0,1]$ into disjoint equivalence classes. Let $H \subset [0,1]$ be the set consisting of exactly one element from each of the equivalence classes. Show that H is uncountable. *Hint: Use the result from the previous problem.*

4. [5 points] Suppose $\Omega = [0,1]$, the interval containing all non-negative real numbers less than or equal to 1. Suppose $\mathcal{F}$ is the set of subsets A of Ω such that either A or A^c is finite. Let $P : \mathcal{F} \mapsto [0,1]$ be defined by $P(A) = 0$ if A is finite and $P(A) = 1$ if A^c is finite. Answer the following with **justification**.

   (a) Is $\mathcal{F}$ a field?

   (b) Is $\mathcal{F}$ a σ-field?

   (c) Is P finitely additive?

   (d) Is P countably additive?