EE 325: Probability and Random Processes (Spring 2025)
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Assignment 4: 20 points — Date: March 22, 2025

1. Let X be a Gaussian random variable with mean $\mu = -3$ and variance $\sigma^2 = 4$. Express the folllowing probabilities in terms of the Q function with positive arguments.
   (a) [1 point] $P[X > 5]$
   (b) [1 point] $P[X < -1]$
   (c) [1 point] $P[1 < X < 4]$
   (d) [2 points] $P[X^2 + X > 2]$

2. [5 points] Suppose $X \sim \mathcal{N}(0, 1)$ and W is equally likely to be +1 or -1. Assume that W is independent of X. If $Y = WX$, prove the following statements.
   (a) $Y \sim \mathcal{N}(0, 1)$.
   (b) X and Y are uncorrelated.
   (c) X and Y are not independent.

3. [5 points] Consider the following ternary hypothesis testing problem where the hypotheses are equally likely.
$$\begin{aligned} H_1 &: Y \sim \mathcal{N}(-\mu, \sigma^2) \\ H_2 &: Y \sim \mathcal{N}(0, \sigma^2) \\ H_3 &: Y \sim \mathcal{N}(\mu, \sigma^2) \end{aligned}$$
   Here Y is the observation and $\mathcal{N}(\mu, \sigma^2)$ denotes the Gausssian distribution with mean μ and variance σ^2.
   (a) Find the optimal decision rule which minimizes the decision error probability. Simplify it as much as possible.
   (b) Find the decision error probability of the optimal decision rule in terms of the Q function.

4. [5 points] Suppose X and Y are jointly Gaussian random variables. Let the joint pdf be given by
$$p_{XY}(x, y) = \frac{1}{2\pi|\mathbf{C}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{s} - \boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{s} - \boldsymbol{\mu})\right)$$
   where $\mathbf{s} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\boldsymbol{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$.
   Suppose Y is observed and we want to estimate X. Derive the MMSE estimator of X.