EE 325: Probability and Random Processes (Spring 2025)
Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay

Midsemester Exam: 40 points March 1, 2025

1. (5 points) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Events are elements of the σ-field $\mathcal{F}$.
   (a) Let $A_1, A_2, \ldots$ be an increasing sequence of events, so that $A_1 \subseteq A_2 \subseteq \cdots$. Let A be their limit
   $$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i.$$
   Prove that $P(A) = \lim_{i\to\infty} P(A_i)$.
   (b) Let $B_1, B_2, \ldots$ be a decreasing sequence of events, so that $B_1 \supseteq B_2 \supseteq \cdots$. Let B be their limit
   $$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i.$$
   Prove that $P(B) = \lim_{i\to\infty} P(B_i)$.

2. (5 points) Consider the Monty Hall problem with four doors. One of the doors has a car behind it and the other three have goats. The car is equally likely to be behind any of the four doors. A contestant picks a door at random. The game show host then reveals one of the other doors which do not have the car. If the contestant always switches from his currently chosen door to one of the two doors which are not open, find the probability that he wins the car. Assume that both the host and contestant choose randomly when faced with multiple choices for doors. ***Note:*** *You have to show the steps used to arrive at the answer. Just stating the final answer is not enough.*

3. (5 points) An urn contains n tickets numbered 1 to n. Two tickets are drawn without replacement. Let X denote the smaller and Y the larger of the two numbers so obtained.
   (a) Find the joint probability mass function of X and Y.
   (b) Find the marginal probability mass functions of X and Y.

4. (5 points) Show that the sum of two independent binomial random variables, $\text{bin}(m, p)$ and $\text{bin}(n, p)$ respectively, is $\text{bin}(m+n, p)$. The notation $\text{bin}(n, p)$ represents a binomial random variable with parameters n and p.

5. (5 points) I am selling my house, and have decided to accept the first offer exceeding ₹K. Assuming that the offers are independent random variables with common distribution function F, find the expected number of offers received before I sell my house. ***Note:*** *You have to show some steps explaining how you arrived at the solution.*

6. (5 points) Let X be a continuous random variable with probability density function given by
   $$f_X(x) = \lambda e^{-\lambda x}, \qquad 0 < x < \infty$$
   where $\lambda > 0$. Find $E[X|X > 1]$. *Hint: You need the conditional probability density function of X given $X > 1$. What is the conditional distribution function of X given $X > 1$?*

7. (5 points) The coefficient of correlation $\rho(X, Y)$ between random variables X and Y is defined as
   $$\rho(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sigma(X)\sigma(Y)}$$
   where $\sigma(X)$ and $\sigma(Y)$ are the standard deviations of X and Y respectively. You can assume that $\sigma(X) \neq 0$ and $\sigma(Y) \neq 0$. Show that $\rho(X, Y)$ always lies in the interval $[-1, 1]$.

8. (5 points) If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent Gaussian random variables, show that $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$. ***Hint:*** *It is enough to show that $X_1 + X_2 - \mu_1 - \mu_2 \sim \mathcal{N}(0, \sigma_1^2 + \sigma_2^2)$.*