EE 325: Probability and Random Processes (Spring 2025)
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Quiz 2 : 30 points April 4, 2025

1. Let X be a Gaussian random variable with mean $\mu = -3$ and variance $\sigma^2 = 9$. Express the folllowing probabilities in terms of the Q function with positive arguments.
   (a) (1 point) $P[X > 6]$
   (b) (1 point) $P[X < -6]$
   (c) (1 point) $P[3 < X < 6]$
   (d) (2 points) $P[X^2 + 3X > 18]$

2. (5 points) Suppose $Y \sim \mathcal{N}(0, 1)$ and Z is equally likely to be $+1$ or -1. Assume that Z is independent of Y. If $X = ZY$, prove the following statements.
   (a) $X \sim \mathcal{N}(0, 1)$.
   (b) Y and X are uncorrelated.
   (c) Y and X are not independent.

3. (5 points) Consider the following ternary hypothesis testing problem where the hypotheses are equally likely.
$$\begin{aligned} H_1 &: Y \sim \mathcal{N}(-\mu, \sigma^2) \\ H_2 &: Y \sim \mathcal{N}(0, \sigma^2) \\ H_3 &: Y \sim \mathcal{N}(\mu, \sigma^2) \end{aligned}$$
   Here Y is the observation and $\mathcal{N}(\mu, \sigma^2)$ denotes the Gausssian distribution with mean μ and variance σ^2.
   (a) Find the optimal decision rule which minimizes the decision error probability. Simplify it as much as possible.
   (b) Find the decision error probability of the optimal decision rule in terms of the Q function.

4. (5 points) Suppose we observe Y_i, $i = 1, 2, \ldots, M$ such that
$$Y_i \sim \text{Uniform}\left[-\frac{\theta}{2}, 2\theta\right]$$
   where Y_i's are independent and θ is unknown. Assume $\theta \geq 0$. Derive the maximum likelihood estimator of θ.

5. Suppose you are given a random variable U which is uniformly distributed in the unit interval $[0, 1]$, i.e. $U \sim \mathcal{U}[0, 1]$. Describe procedures to generate the following random variables using **only** U.
   (a) ($2\frac{1}{2}$ points) An exponential random variable X with parameter $\lambda > 0$ that has distribution function
$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0.$$
   (b) ($2\frac{1}{2}$ points) A binomial random variable Y with parameters n and p whose probability mass function is given by
$$P[Y = k] = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{if } 0 \leq k \leq n.$$

6. (5 points) Suppose we want to generate a random variable X having probability density function (pdf) f. Suppose X is difficult to generate using the inversion method. Suppose there is a random variable Y with pdf g which is easy to generate using the inversion method. For some $c \in \mathbb{R}$, suppose f and g satisfy
$$\frac{f(y)}{cg(y)} \leq 1 \text{ for all } y.$$
   We use the following procedure.
   - Generate a uniform random variable $U \sim \mathcal{U}[0, 1]$.
   - Generate the random variable Y using the inversion method. Note that this method uses a different uniform random variable, i.e. it does not use U.
   - If $U \leq \frac{f(Y)}{cg(Y)}$, set $X = Y$. Otherwise, generate another pair (U, Y) and keep trying until the inequality is satisfied.

   Prove that the X which is output by the above procedure is a random variable with probability density function f.