Gaussian Random Variables

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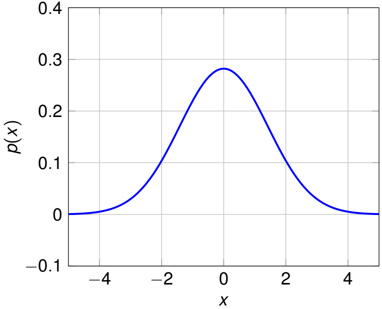
Gaussian Random Variable

Definition

A continuous random variable with pdf of the form

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < \infty,$$

where μ is the mean and σ^2 is the variance.



Notation

- $\mathcal{N}(\mu, \sigma^2)$ denotes a Gaussian distribution with mean μ and variance σ^2
- $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow X$ is a Gaussian RV with mean μ and variance σ^2
- If $X \sim \mathcal{N}(0, 1)$, then X is a standard Gaussian RV

Affine Transformations Preserve Gaussianity

Theorem

If X is Gaussian, then $aX + b$ is Gaussian for $a, b \in \mathbb{R}$.

Remarks

- If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$ and $\sigma \neq 0$, then $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$.

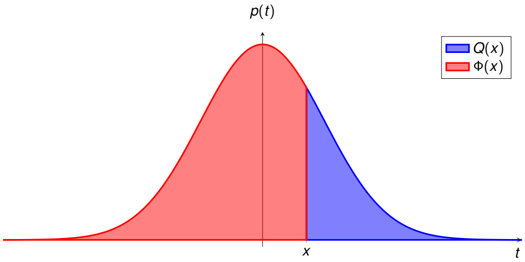
CDF and CCDF of Standard Gaussian

- Cumulative distribution function of $X \sim \mathcal{N}(0, 1)$

$$\Phi(x) = P[X \leq x] = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

- Complementary cumulative distribution function of $X \sim \mathcal{N}(0, 1)$

$$Q(x) = P[X > x] = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$



Properties of $Q(x)$

- $\Phi(x) + Q(x) = 1$
- $Q(-x) = \Phi(x) = 1 - Q(x)$
- $Q(0) = \frac{1}{2}$
- $Q(\infty) = 0$
- $Q(-\infty) = 1$
- $X \sim \mathcal{N}(\mu, \sigma^2)$

$$P[X > \alpha] = Q\left(\frac{\alpha - \mu}{\sigma}\right)$$

$$P[X \leq \alpha] = Q\left(\frac{\mu - \alpha}{\sigma}\right)$$

Jointly Gaussian Random Variables

Jointly Gaussian Random Variables

Definition (Jointly Gaussian RVs)

Random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if any linear combination is a Gaussian random variable.

$$a_1 X_1 + \cdots + a_n X_n \text{ is Gaussian for all } (a_1, \ldots, a_n) \in \mathbb{R}^n.$$

Example (Not Jointly Gaussian)

$X \sim \mathcal{N}(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim \mathcal{N}(0, 1)$ and $X + Y$ is not Gaussian.

Covariance

- For real random variables X and Y, the covariance is defined as

$$\mathrm{cov}(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right]$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$
- Properties
  - $\mathrm{var}(X) = \mathrm{cov}(X, X)$
  - If X and Y are independent, then $\mathrm{cov}(X, Y) = 0$
  - If $\mathrm{cov}(X, Y) = 0$, then they are said to be uncorrelated
  - $\mathrm{cov}(X + a, Y + b) = \mathrm{cov}(X, Y)$ for any $a, b \in \mathbb{R}$
  - Covariance is a bilinear function

$$\begin{aligned}\mathrm{cov}\left(a_1 X_1 + a_2 X_2, a_3 X_3 + a_4 X_4\right) &= a_1 a_3 \,\mathrm{cov}(X_1, X_3) \\ &\quad + a_1 a_4 \,\mathrm{cov}(X_1, X_4) \\ &\quad + a_2 a_3 \,\mathrm{cov}(X_2, X_3) \\ &\quad + a_2 a_4 \,\mathrm{cov}(X_2, X_4)\end{aligned}$$

  - Correlation coefficient of X and Y is defined as

$$\rho(X, Y) = \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}}.$$

  - $|\rho(X, Y)| \le 1$ with equality $\iff$ $\Pr\left[aY = bX + c\right] = 1$ for some constants a, b, c

Mean Vector and Covariance Matrix

- Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a $n \times 1$ random vector
- The mean vector of $\mathbf{X}$ is given by $\mathbf{m}_X = E[\mathbf{X}] = (E[X_1], \ldots, E[X_n])^T$
- The covariance matrix $\mathbf{C}_X$ of $\mathbf{X}$ is an $n \times n$ matrix with (i, j)th entry given by

$$\begin{aligned} \mathbf{C}_X(i,j) &= E\left[(X_i - E[X_i])(X_j - E[X_j])\right] \\ &= E[X_iX_j] - E[X_i]E[X_j] \end{aligned}$$

- A compact notation for $\mathbf{C}_X$ is

$$\mathbf{C}_X = E\left[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T\right] = E\left[\mathbf{X}\mathbf{X}^T\right] - E[\mathbf{X}]\left(E[\mathbf{X}]\right)^T$$

- If $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where $\mathbf{A}$ is $m \times n$ constant matrix and $\mathbf{b}$ is an $m \times 1$ constant vector, then

$$\begin{aligned} \mathbf{m}_Y &= \mathbf{A}\mathbf{m}_X + \mathbf{b} \\ \mathbf{C}_Y &= \mathbf{A}\mathbf{C}_X\mathbf{A}^T \end{aligned}$$

Gaussian Random Vector

Definition (Gaussian Random Vector)

A random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ whose components are jointly Gaussian.

Notation

$\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C})$ where

$$\mathbf{m} = E[\mathbf{X}], \quad \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T\right]$$

Definition (Joint Gaussian Density)

If $\mathbf{C}$ is invertible, the joint density is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})\right)$$

Example ($\mathbf{C}$ is not invertible)

$\mathbf{X} = (X_1, X_2)^T$ where $X_1 \sim \mathcal{N}(0, 1)$ and $X_2 = 2X_1 + 3$

Affine Transformations Preserve Joint Gaussianity

- If $\mathbf{X}$ is a Gaussian vector, then $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ is also a Gaussian vector
  - Here $\mathbf{X}$ is an $n \times 1$ vector, $\mathbf{A}$ is an $m \times n$ constant matrix, and $\mathbf{b}$ is an $m \times 1$ constant vector
  - Any linear combination of $Y_1, \ldots, Y_m$ is a constant plus a linear combination of $X_1, \ldots, X_n$, which is a Gaussian random variable
- Since $\mathbf{Y}$ is a Gaussian random vector, its distribution is completely characterized by its mean vector and covariance matrix

$$\mathbf{X} \sim \mathcal{N}(\mathbf{m}, \mathbf{C}) \implies \mathbf{Y} \sim \mathcal{N}\left(\mathbf{Am} + \mathbf{b}, \mathbf{ACA}^T\right)$$

Uncorrelated Random Variables and Independence

- Recall that X_1 and X_2 are said to be uncorrelated if $\mathrm{cov}(X_1, X_2) = 0$
- If X_1 and X_2 are independent,

$$\mathrm{cov}(X_1, X_2) = 0.$$

- If $X_1, \ldots, X_n$ are jointly Gaussian and pairwise uncorrelated, then they are independent. Consider the case when $\mathrm{var}(X_i) \neq 0$ for each i.

$$\begin{aligned} p(\mathbf{x}) &= \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left(-\frac{(x_i - m_i)^2}{2\sigma_i^2}\right) \end{aligned}$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \mathrm{var}(X_i)$.

Uncorrelated Gaussian RVs may not be Independent

Example

- $X \sim \mathcal{N}(0, 1)$
- W is equally likely to be +1 or -1
- W is independent of X
- $Y = WX$
- $Y \sim \mathcal{N}(0, 1)$
- X and Y are uncorrelated
- X and Y are not independent

References

- Section 3.1, *Fundamentals of Digital Communication*, Upamanyu Madhow, 2008
- Dirac delta function `https://en.wikipedia.org/wiki/Dirac_delta_function`