Gaussian Random Processes

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Gaussian Random Process

Definition

A random process $X(t)$ is Gaussian if its samples $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for any $n \in \mathbb{N}$ and distinct sample locations $t_1, t_2, \ldots, t_n$.

Let $\mathbf{X} = \begin{bmatrix} X(t_1) & \cdots & X(t_n) \end{bmatrix}^T$ be the vector of samples. The joint density is given by

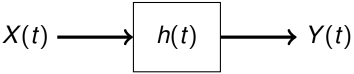
$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m}) \right)$$

where

$$\mathbf{m} = E[\mathbf{X}], \;\; \mathbf{C} = E\left[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T \right]$$

Properties of Gaussian Random Process

- The mean and autocorrelation functions completely characterize a Gaussian random process.
- Wide-sense stationary Gaussian processes are strictly stationary.
- If the input to a stable linear filter is a Gaussian random process, the output is also a Gaussian random process.



White Gaussian Noise

Definition

A zero mean WSS Gaussian random process with constant power spectral density

$$S_n(f) = \frac{N_0}{2}.$$

$\frac{N_0}{2}$ is termed the two-sided PSD and has units Watts per Hertz.

Remarks

- Autocorrelation function $R_n(\tau) = \frac{N_0}{2}\delta(\tau)$
- Infinite Power! Ideal model of Gaussian noise occupying more bandwidth than the signals of interest.

White Gaussian Noise through Correlators

- Consider the output of a correlator with WGN input

$$Z = \int_{-\infty}^{\infty} n(t)u(t)\, dt = \langle n, u \rangle$$

where $u(t)$ is a deterministic finite-energy signal

- Z is a Gaussian random variable
- The mean of Z is

$$E[Z] = \int_{-\infty}^{\infty} E\left[n(t)\right] u(t)\, dt = 0$$

- The variance of Z is

$$\begin{aligned}
\operatorname{var}(Z) &= E\left[(\langle n, u \rangle)^2\right] = E\left[\int n(t)u(t)\, dt \int n(s)u(s)\, ds\right] \\
&= \int \int u(t)u(s) E\left[n(t)n(s)\right]\, dt\, ds \\
&= \int \int u(t)u(s) \frac{N_0}{2} \delta(t-s)\, dt\, ds \\
&= \frac{N_0}{2} \int u^2(t)\, dt = \frac{N_0}{2} \|u\|^2
\end{aligned}$$

White Gaussian Noise through Correlators

Proposition

Let $u_1(t)$ and $u_2(t)$ be linearly independent finite-energy signals and let $n(t)$ be WGN with PSD $S_n(f) = \frac{N_0}{2}$. Then $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian with covariance

$$\text{cov}\left(\langle n, u_1 \rangle, \langle n, u_2 \rangle\right) = \frac{N_0}{2} \langle u_1, u_2 \rangle.$$

Proof

To prove that $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian, consider a non-trivial linear combination $a\langle n, u_1 \rangle + b\langle n, u_2 \rangle$

$$a\langle n, u_1 \rangle + b\langle n, u_2 \rangle = \int n(t) \left[a u_1(t) + b u_2(t)\right] \, dt.$$

This is the result of passing $n(t)$ through a correlator. So it is a Gaussian random variable.

White Gaussian Noise through Correlators

Proof (continued)

$$\begin{aligned}
\operatorname{cov}\left(\langle n, u_1\rangle, \langle n, u_2\rangle\right) &= E\left[\langle n, u_1\rangle\langle n, u_2\rangle\right] \\
&= E\left[\int n(t)u_1(t)\,dt \int n(s)u_2(s)\,ds\right] \\
&= \int\int u_1(t)u_2(s)E\left[n(t)n(s)\right]\,dt\,ds \\
&= \int\int u_1(t)u_2(s)\frac{N_0}{2}\delta(t-s)\,dt\,ds \\
&= \frac{N_0}{2}\int u_1(t)u_2(t)\,dt \\
&= \frac{N_0}{2}\langle u_1, u_2\rangle
\end{aligned}$$

If $u_1(t)$ and $u_2(t)$ are orthogonal, $\langle n, u_1\rangle$ and $\langle n, u_2\rangle$ are independent.

Reference

- Chapter 3, *Fundamentals of Digital Communication*, Upamanyu Madhow, Cambridge University Press, 2008.