# Limit Theorems

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# Limit Theorems

## Theorem (Weak Law of Large Numbers)

*Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means $\mu$. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy*

$$\frac{S_n}{n} \xrightarrow{P} \mu \qquad \text{as } n \to \infty.$$

## Theorem (Central Limit Theorem)

*Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means $\mu$ and finite non-zero variance $\sigma^2$. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy*

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0, 1) \qquad \text{as } n \to \infty.$$

# Characteristic Functions

# Characteristic Functions

## Definition

For a random variable $X$, the characteristic function is given by

$$\phi(t) = E(e^{itX})$$

## Examples

- **Bernoulli RV:** $P(X = 1) = p$ and $P(X = 0) = 1 - p$

$$\phi(t) = 1 - p + pe^{it} = q + pe^{it}$$

- **Gaussian RV:** Let $X \sim N(\mu, \sigma^2)$

$$\phi(t) = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$$

# Properties of Characteristic Functions

## Theorem

*If $X$ and $Y$ are independent, then*

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(s).$$

## Example (Binomial RV)

$$\phi(t) = \left(q + pe^{it}\right)^n$$

## Example (Sum of Independent Gaussian RVs)

Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent. What is the distribution of $X + Y$?

## Theorem

*If $a, b \in \mathbb{R}$ and $Y = aX + b$, then*

$$\phi_Y(t) = e^{itb}\phi_X(at).$$

# Inversion and Continuity Theorems

## Theorem

*Random variables $X$ and $Y$ have the same characteristic function if and only if they have the same distribution function.*

## Definition

We say that the sequence $F_1, F_2, \ldots$ of distribution functions **converges** to a distribution function $F$, written as $F_n \to F$, if $\lim_{n\to\infty} F_n(x) = F(x)$ at each point $x$ where $F$ is continuous.

## Theorem

*Suppose $F_1, F_2, \ldots$ is a sequence of distribution functions with corresponding characteristic functions $\phi_1, \phi_2, \ldots$.*

- *If $F_n \to F$ for some distribution function $F$ with characteristic function $\phi$, then $\phi_n(t) \to \phi(t)$ for all $t$.*
- *Conversely, if $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists and is continuous at $t = 0$, then $\phi$ is the characteristic function of some distribution function $F$, and $F_n \to F$.*

# Limit Theorems

# Weak Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means $\mu$. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{S_n}{n} \xrightarrow{P} \mu \qquad \text{as } n \to \infty.$$

## Proof.

- Since $\mu$ is a constant, it is enough to show convergence in distribution
- It is enough to show that the characteristic functions of $\frac{S_n}{n}$ converge to the characteristic function of $\mu$
- By Taylor's theorem, the characteristic function of the $X_n$'s is

$$\phi_{X_n}(t) = E\left[e^{itX_n}\right] = 1 + i\mu t + o(t)$$

- The characteristic function of $\frac{S_n}{n}$ is

$$\phi_n(t) = \left[\phi_{X_1}\left(\frac{t}{n}\right)\right]^n = \left[1 + i\mu\frac{t}{n} + o\left(\frac{t}{n}\right)\right]^n \to \exp(it\mu)$$

# Strong Law of Large Numbers

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables. Then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to \mu \quad \text{almost surely, as } n \to \infty.$$

for some constant $\mu$, if and only if $E|X_1| < \infty$. In this case, $\mu = E[X_1]$.

# Central Limit Theorem

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means $\mu$ and finite non-zero variance $\sigma^2$. Their partial sums $S_n = X_1 + X_2 + \cdots + X_n$ satisfy

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{D} N(0,1) \qquad \text{as } n \to \infty.$$

## Proof.

- It is enough to show that the characteristic functions of $\frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ converge to the characteristic function of $Z \sim N(0,1)$ which is $e^{-\frac{t^2}{2}}$
- Let $\phi_Y(t)$ be the characteristic function of $Y_n = \frac{X_n - \mu}{\sigma}$
- By Taylor's theorem, the characteristic function of the $Y_n$'s is

$$\phi_Y(t) = E\left[e^{itY}\right] = 1 - \frac{t^2}{2} + o(t^2)$$

- The characteristic function of $\frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} Y_j$ is

$$\psi_n(t) = \left[\phi_Y\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]^n \to \exp\left(-\frac{t^2}{2}\right)$$

# Reference

- Chapter 5, *Probability and Random Processes*, Grimmett and Stirzaker, Fourth Edition, 2020.