

Power Spectral Density

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Power Spectral Density

- Fourier transform

$$\begin{aligned}X(f) &= \int_{-\infty}^{\infty} x(t) \exp(-j2\pi ft) dt \\X(f) &= \mathcal{F}(x(t))\end{aligned}$$

- Inverse Fourier transform

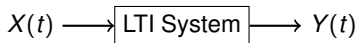
$$\begin{aligned}x(t) &= \int_{-\infty}^{\infty} X(f) \exp(j2\pi ft) df \\x(t) &= \mathcal{F}^{-1}(X(f))\end{aligned}$$

Definition (Power Spectral Density of a WSS Process)

The power spectral density of a wide-sense stationary random process is the Fourier transform of the autocorrelation function.

$$S_X(f) = \mathcal{F}(R_X(\tau))$$

Motivating the Definition of Power Spectral Density



- Consider an LTI system with impulse response $h(t)$ which has random processes $X(t)$ and $Y(t)$ as input and output

$$Y(t) = \int_{-\infty}^{\infty} h(\tau) X(t - \tau) d\tau$$

- If $X(t)$ is a wide-sense stationary random process, then $Y(t)$ is also wide-sense stationary with autocorrelation function

$$R_Y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau - \tau_1 + \tau_2) d\tau_1 d\tau_2$$

- Setting $\tau = 0$, we can express the average power in the output process as

$$R_Y(0) = E[Y^2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2$$

Motivating the Definition of Power Spectral Density

- Let $H(f)$ be the Fourier transform of the impulse response $h(t)$

$$h(\tau_1) = \int_{-\infty}^{\infty} H(f) \exp(j2\pi f\tau_1) df$$

- Substituting the above equation into the average power equation we get

$$\begin{aligned} E[Y^2(t)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1) h(\tau_2) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(f) e^{j2\pi f\tau_1} df \right] h(\tau_2) R_X(\tau_2 - \tau_1) d\tau_1 d\tau_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(f) \left[\int_{-\infty}^{\infty} h(\tau_2) e^{j2\pi f\tau_2} d\tau_2 \right] R_X(\tau) e^{-j2\pi f\tau} d\tau df \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(f) H^*(f) R_X(\tau) e^{-j2\pi f\tau} d\tau df \\ &= \int_{-\infty}^{\infty} |H(f)|^2 \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau df \\ &= \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df \end{aligned}$$

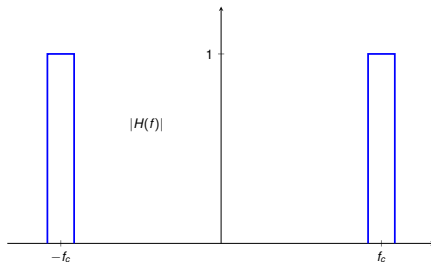
Motivating the Definition of Power Spectral Density

- The output power and power spectral density are related by

$$E \left[Y^2(t) \right] = \int_{-\infty}^{\infty} |H(f)|^2 S_X(f) df$$

- Let the LTI system be an ideal narrowband filter with magnitude response given by

$$|H(f)| = \begin{cases} 1 & |f \pm f_c| \leq \frac{\Delta f}{2} \\ 0 & |f \pm f_c| > \frac{\Delta f}{2} \end{cases}$$



$$E \left[Y^2(t) \right] \approx (2\Delta f) S_X(f_c)$$

Properties of Power Spectral Density

- The power spectral density and autocorrelation function form a Fourier transform pair

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-i2\pi f\tau) d\tau$$
$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) \exp(i2\pi f\tau) df$$

- Power spectral density is a non-negative and even function of f
- Power of $X(t)$ equals area under power spectral density

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f) df$$

- If $X(t)$ is passed through an LTI system with frequency response $H(f)$ to get $Y(t)$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

White Noise

- A wide-sense stationary random process with flat power spectral density

$$S_W(f) = c$$

where c has units Watts per Hertz

- White noise has infinite power and is not physically realizable
- Models a situation where the noise bandwidth is much larger than the signal bandwidth
- The corresponding autocorrelation function is given by

$$R_N(\tau) = c\delta(\tau)$$

where δ is the Dirac delta function

Reference

- Chapter 1, *Communication Systems*, Simon Haykin, Fourth Edition, Wiley-India, 2001.