Properties of Probability Measures

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Probability Space

Definition

A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of a set Ω, a σ-field $\mathcal{F}$ of subsets of Ω and a probability measure P on $(\Omega, \mathcal{F})$.

Definition

A probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ satisfying

(a) $P(\Omega) = 1$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of **disjoint** sets in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Some Properties of Probability Measures

- $P(\phi) = 0$
- For a disjoint collection $A_1, A_2, \ldots A_n \in \mathcal{F}$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

- $P(A^c) = 1 - P(A)$
- Define $B \setminus A = B \cap A^c$. If $A \subseteq B$, then $P(B) = P(A) + P(B \setminus A) \geq P(A)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- For any $n \in \mathbb{N}$

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_{i} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \\ &\cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots A_n) \end{aligned}$$

(**inclusion-exclusion principle**)

P is a continuous set function

Theorem

Let $A_1, A_2, \ldots$ be an increasing sequence of events, so that $A_1 \subseteq A_2 \subseteq \cdots$. Let A be their limit

$$A = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i.$$

Then $P(A) = \lim_{i \to \infty} P(A_i)$.

Proof.

The set A can be written as a disjoint union as follows

$$A = A_1 \bigcup (A_2 \setminus A_1) \bigcup (A_3 \setminus A_2) \bigcup (A_4 \setminus A_3) \cdots$$

By the countable additivity property of P, we have

$$P(A) = P(A_1) + \sum_{i=1}^{\infty} P(A_{i+1} \setminus A_i) = P(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n} P(A_{i+1} \setminus A_i)$$

$$= P(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n} [P(A_{i+1}) - P(A_i)] = P(A_1) + \lim_{n \to \infty} [P(A_{n+1}) - P(A_1)]$$

$$= \lim_{n \to \infty} P(A_{n+1})$$

P is a continuous set function

Theorem

Let $B_1, B_2, \ldots$ be a decreasing sequence of events, so that $B_1 \supseteq B_2 \supseteq \cdots$. Let B be their limit

$$B = \bigcap_{i=1}^{\infty} B_i = \lim_{i \to \infty} B_i.$$

Then $P(B) = \lim_{i \to \infty} P(B_i)$.

Proof.

Let $A_i = B_i^c$ and use the previous theorem.

Some Jargon

- Null events
  - An event A is called **null** if $P(A) = 0$
  - Null events should not be confused with the impossible event ϕ
  - The impossible event is null, but null events need not be impossible
- Events that occur almost surely
  - An event A is said to occur **almost surely** if $P(A) = 1$
  - Frequently abbreviated to a.s.
  - The certain event occurs almost surely, but every event that occurs almost surely is not the certain event

Reference

- Section 1.3 from *Probability and Random Processes*, G. Grimmett and D. R. Stirzaker, 2001 (3rd Edition)