Probability Spaces

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Probability Theory

- Mathematical theory of uncertainty
- Complete information is difficult to obtain in many situations
  - Toss of a coin
  - Customer arrivals at a bank
- Probability theory gives us tools to analyze such situations
- Applications
  - Communications and signal processing
  - Physics
  - Banking and Finance
  - Gambling

What is Probability?

- Informally, a method of quantifying the degree of certainty of a situation
- Classical definition: Ratio of favorable outcomes and the total number of outcomes provided all outcomes are equally likely.

$$P(A) = \frac{N_A}{N}$$

- Relative frequency definition:

$$P(A) = \lim_{N \to \infty} \frac{N_A}{N}$$

- Axiomatic definition: A countably additive function defined on the set of events with range in the interval $[0, 1]$.
- The axiomatic definition will be used in this course

Sample Space

The first step in constructing a probabilistic model for a situation is to list all the possible outcomes

Definition

The set of all possible outcomes of an experiment is called the sample space and is denoted by Ω.

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Tossing of two coins: $\Omega = \{(H, H), (T, H), (H, T), (T, T)\}$
- Coin is tossed until heads appear. What is Ω?
- Life expectancy of a random person. $\Omega = [0, 120]$ years

Events

- An event is a subset of the sample space
- An event is said to have occurred if the outcome of the experiment belongs to it

Examples

- Coin toss: $\Omega = \{\text{Heads}, \text{Tails}\}$.
  $E = \{\text{Heads}\}$ is the event that a head appears on the flip of a coin.
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
  $E = \{2, 4, 6\}$ is the event that an even number appears.
- Life expectancy. $\Omega = [0, 120]$.
  $E = [50, 120]$ is the event that a random person lives beyond 50 years.

Language of Events

Typical Notation	Language of Sets	Language of Events
Ω	Whole space	Certain event
ϕ	Empty set	Impossible event
A	Subset of Ω	Event that some outcome in A occurs
A^c	Complement of A	Event that no outcome in A occurs
$A \cup B$	Union	Event that an outcome in A or B or both occurs
$A \cap B$	Intersection	Event that an outcome in both A and B occurs
$A \cap B = \phi$	Disjoint sets	Mutually exclusive events

Assigning Probabilities to Events

- We want to assign probabilities to events
    - Coin toss: $\Omega = \{\mathrm{H}, \mathrm{T}\}$

$$P(\phi) = 0, \;\; P(\mathrm{H}) = \frac{1}{2}, \;\; P(\mathrm{T}) = \frac{1}{2}, \;\; P(\Omega) = 1$$

    - Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$P(A) = \frac{|A|}{6} \text{ for any } A \subseteq \Omega$$

- Can we always assign probabilities consistently to all the subsets of a sample space?
    - Yes, if the sample space is finite or countable
    - Not always, if the sample space is uncountable (example in next lecture)

Which subsets must be events?

- Let $\mathcal{F}$ be a subset of the power set 2^Ω consisting of events to which we will assign probabilities
- If $\mathcal{F} \neq 2^\Omega$, which subsets of Ω must be there in $\mathcal{F}$?
  - If we are interested in an event A, then A^c is also interesting

$$A \in \mathcal{F} \implies A^c \in \mathcal{F}$$

  - If events A and B are interesting, then their simultaneous occurrence is also interesting

$$A, B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$$

  - These two requirements give us the following (Why?)

$$A, B \in \mathcal{F} \implies A \cup B \in \mathcal{F}$$

  - They also give us $\phi \in \mathcal{F}$ if $\mathcal{F}$ is nonempty (Why?)
- Any $\mathcal{F}$ which satisfies these conditions is called a field
- To deal with infinite sample spaces, $\mathcal{F}$ needs to be a σ-field

σ-fields

Definition

A collection $\mathcal{F}$ of subsets of Ω is called a σ-field if it satisfies

(a) $\phi \in \mathcal{F}$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

(c) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Examples

- $\mathcal{F} = \{\phi, \Omega\}$ is the smallest σ-field
- If $A \subseteq \Omega$, $\mathcal{F} = \{\phi, A, A^c, \Omega\}$ is a σ-field
- 2^{Ω} is a σ-field

Exercises

- Is $\mathcal{F} = \left\{ A \subseteq \mathbb{N} \,\middle|\, A \text{ is a finite set} \right\}$ a σ-field?
- Is $\mathcal{F} = \left\{ A \subseteq \mathbb{N} \,\middle|\, \text{Either } A \text{ or } A^c \text{ is finite} \right\}$ a σ-field?

Probability Measure

Definition

Let $\mathcal{F}$ be a σ-field of subsets of Ω. A probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \mapsto [0, 1]$ satisfying

(a) $P(\Omega) = 1$

(b) if $A_1, A_2, \ldots \in \mathcal{F}$ is a collection of disjoint members in $\mathcal{F}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(P is said to be countably additive)

Examples

- Coin toss: $\Omega = \{\mathrm{H}, \mathrm{T}\}$, $\mathcal{F} = \{\phi, \mathrm{H}, \mathrm{T}, \Omega\}$

$$P(\phi) = 0, \;\; P(\mathrm{H}) = p, \;\; P(\mathrm{T}) = 1 - p, \;\; P(\Omega) = 1$$

- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$, $P(\{i\}) = p_i$ for $i = 1, \ldots, 6$, $\sum_{i=1}^{6} p_i = 1$.

$$P(A) = \sum_{i \in A} p_i \text{ for any } A \subseteq \Omega$$

Probability Space

Definition

A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of

- a set Ω,
- a σ-field $\mathcal{F}$ of subsets of Ω and
- a probability measure P on $(\Omega, \mathcal{F})$.

Summary

- Probability theory is the mathematical theory of uncertainty
- The axiomatic definition will be used in this course
- Set of all possible outcomes is called the sample space Ω
- An event is a subset of the sample space
- The set of events is a σ-field $\mathcal{F}$
- A probability measure is a countably additive set function $P : \mathcal{F} \mapsto [0, 1]$
- A probability space is a triple $(\Omega, \mathcal{F}, P)$

References

- Sections 1.1, 1.2, 1.3 from Grimmett and Stirzaker