Random Variables

Saravanan Vijayakumaran

Department of Electrical Engineering Indian Institute of Technology Bombay

January 22, 2025

Measurements in Experiments

- In many experiments, we are interested in some real-valued measurement
- Example
 - A coin is tossed twice. We want to count the number of heads which appear.
 - $\Omega = \{HH, HT, TH, TT\}$
 - Let $X(\omega)$ be the number of heads for $\omega \in \Omega$.
 - X(HH) = 2, X(HT) = 1, X(TH) = 1, X(TT) = 0
- We are also interested in knowing which measurements are more likely and which are less likely
- The distribution function $F : \mathbb{R} \to [0, 1]$ captures this information where

 $F(x) = Probability that X(\omega) is less than or equal to x$ $= P(\{\omega \in \Omega : X(\omega) \le x\})$

Is {ω ∈ Ω : X(ω) ≤ x} always an event? Does it always belong to the σ-field *F* of the experiment?

Random Variables

Definition (Random Variable)

A random variable is a function $X : \Omega \to \mathbb{R}$ with the property that $\{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{F}$ for each $x \in \mathbb{R}$.

Definition (Distribution Function)

The distribution function of a random variable X is the function $F : \mathbb{R} \to [0, 1]$ given by $F(x) = P(X \le x)$

Examples

- Counting heads in two tosses of a coin.
- Constant random variable

$$X(\omega) = c$$
 for all $\omega \in \Omega$

Properties of the Distribution Function

•
$$P(X > x) = 1 - F(x)$$

•
$$P(x < X \le y) = F(y) - F(x)$$

- If x < y, then $F(x) \le F(y)$
- $\lim_{x\to -\infty} F(x) = 0$
- $\lim_{x\to\infty} F(x) = 1$
- *F* is right continuous, $F(x + h) \rightarrow F(x)$ as $h \downarrow 0$

•
$$P(X = x) = F(x) - \lim_{y \uparrow x} F(y)$$

Discrete Random Variables

Discrete Random Variables

Definition

A random variable is called discrete if it takes values only in some countable subset $\{x_1, x_2, x_3, ...\}$ of \mathbb{R} .

Definition

A discrete random variable X has a probability mass function $f : \mathbb{R} \to [0, 1]$ given by f(x) = P[X = x]

Example

Bernoulli random variable

 $\Omega = \{0,1\}$

$$P[X = x] = \begin{cases} p & \text{if } x = 1\\ 1 - p & \text{if } x = 0 \end{cases}$$

where $0 \le p \le 1$

Properties of the Probability Mass Function

Let F be the distribution function and f be the mass function of a random variable

•
$$F(x) = \sum_{i:x_i \leq x} f(x_i)$$

•
$$\sum_{i=1}^{\infty} f(x_i) = 1$$

•
$$f(x) = F(x) - \lim_{y \uparrow x} F(y)$$

Binomial Random Variable

- An experiment is conducted *n* times and it succeeds each time with probability *p* and fails each time with probability 1 − *p*
- The sample space is Ω = {0, 1}ⁿ where 1 denotes success and 0 denotes failure
- Let X denote the total number of successes
- $X \in \{0, 1, 2, \dots, n\}$
- The probability mass function of X is

$$P[X = k] = \binom{n}{k} p^{k} (1 - p)^{n-k} \quad \text{if } 0 \le k \le n$$

- X is said to have the binomial distribution with parameters n and p
- X is the sum of n independent and identically distributed Bernoulli random variables $Y_1 + Y_2 + \cdots + Y_n$

Binomial Random Variable PMF



Poisson Random Variable

- The sample space of a Poisson random variable is $\Omega = \{0, 1, 2, 3, \ldots\}$
- The probability mass function is

$$P[X=k] = \frac{\lambda^k}{k!}e^{-\lambda} \quad k = 0, 1, 2, \dots$$

where $\lambda > 0$

Poisson Random Variable PMF



Independence

- Random variables X and Y are independent if the events $\{X \le x\}$ and $\{Y \le y\}$ are independent for all x and y in \mathbb{R}
- Discrete random variables X and Y are independent if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all x and y in \mathbb{R}
- Examples
 - Two fair dice are rolled. Let *X* be the number shown by the first die and *Y* be the sum of the numbers shown by both dice. Are *X* and *Y* independent?
 - Binary symmetric channel with crossover probability *p*. If the input is equally likely to be 0 or 1, are the input and output independent?
- A family of random variables $\{X_i : i \in I\}$ is an independent family if

$$P\left(\bigcap_{i\in J} \{X_i \leq x_i\}\right) = \prod_{i\in J} P(X_i \leq x_i)$$

for all sets $\{x_i : i \in I\}$ and for all finite subsets $J \subseteq I$

Example

Let *X* and *Y* be independent random variables, each taking values -1 or 1 with equal probability $\frac{1}{2}$. Let *Z* = *XY*. Are *X*, *Y*, and *Z* independent?

Consequences of Independence

- If X and Y are independent, then for any functions $g, h : \mathbb{R} \to \mathbb{R}$ the random variables g(X) and h(Y) are independent
- Exercise
 - Let X and Y be independent discrete random variables taking values in the positive integers
 - · Both of them have the same probability mass function given by

$$P[X = k] = P[Y = k] = \frac{1}{2^k}$$
 for $k = 1, 2, 3, ...$

- Find the following
 - $P(\min\{X, Y\} \leq x)$
 - P[X = Y]
 - P[X > Y]
 - *P*[X ≥ nY] for a given positive integer n
 - P[X divides Y]

Jointly Distributed Discrete Random Variables

Jointly Distributed Discrete Random Variables

Definition

The joint probability distribution function of discrete RVs X and Y is given by

$$F_{X,Y}(x,y) = P\left(X \leq x \bigcap Y \leq y\right).$$

The joint probability mass function is given by

$$f_{X,Y}(x,y) = P\left(X = x \bigcap Y = y\right).$$

Definition

Given the joint pmf, the marginal pmfs are given by

$$f_X(x) = P(X = x) = \sum_{y} f_{X,Y}(x, y)$$
$$f_Y(y) = P(Y = y) = \sum_{x} f_{X,Y}(x, y)$$

Properties of the Joint PMF

- $\sum_{x} \sum_{y} f_{X,Y}(x,y) = 1$
- X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$
 for all $x, y \in \mathbb{R}$

Exercises

- The joint probability mass function of two discrete random variables X and Y is given by f(x, y) = c(2x + y) where x and y take integer values such that 0 ≤ x ≤ 2, 0 ≤ y ≤ 3, and f(x, y) = 0 otherwise. Find the value of c.
- Given independent random variables $X_1, X_2, ..., X_n$ with probability mass functions $f_1, f_2, ..., f_n$ respectively, find the probability mass functions of the following
 - max (X_1, X_2, \ldots, X_n)
 - $\min(X_1, X_2, \ldots, X_n)$

Conditional Distribution

Definition

The conditional probability distribution function of *Y* given X = x is defined as

$$F_{Y|X}(y|x) = P(Y \le y|X = x)$$

for any x such that P(X = x) > 0.

The conditional probability mass function of Y given X = x is defined as

$$f_{Y|X}(y|x) = P(Y = y|X = x)$$

Properties

•
$$\sum_{y} f_{Y|X}(y|x) = 1$$

• $\sum_{x} f_{Y|X}(y|x) f_X(x) = f_Y(y)$

Example

Using the clues given below, find the missing entries in the joint probability mass function of X and Y.

Y/X	1	2	3	
1	?	?	?	-
2	?	0	?	
3	0	?	0	

Table: Joint probability mass function $f_{X,Y}(x, y)$

For k = 1, 2, 3,

- $P(Y = 1 | X = k) = \frac{2}{3}$
- $P(X = k | Y = 1) = \frac{k}{6}$

Sum of Discrete Random Variables

Theorem

For discrete random variables X and Y with joint pmf f(x, y), the pmf of X + Y is given by

$$P(X+Y=z) = \sum_{x} f(x,z-x) = \sum_{y} f(z-y,y)$$

If X and Y are independent, the pmf of X + Y is the convolution of the pmfs of X and Y.

$$P(X + Y = z) = \sum_{x} f_X(x) f_Y(z - x) = \sum_{y} f_X(z - y) f_Y(y)$$

Exercise

X is the sum of *n* independent and identically distributed Bernoulli random variables $Y_1 + Y_2 + \cdots + Y_n$. Can we derive the pmf of *X* using the above identity?

Continuous Random Variables

Continuous Random Variables

Definition

A random variable is called continuous if its distribution function can be expressed as

$$F(x) = \int_{-\infty}^{x} f(u) \, du$$
 for all $x \in \mathbb{R}$

for some integrable function $f : \mathbb{R} \to [0, \infty)$ called the probability density function of *X*. If *F* is differentiable at *u*, then f(u) = F'(u).

Example

Uniform random variable on [0, 1] $\Omega = [0, 1], X(\omega) = \omega, X \sim U[0, 1]$ $f(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ $F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$

Uniform Random Variable on [a, b]

Example

 $X \sim U[a, b]$ $\Omega = [a, b], a < b, X(\omega) = \omega,$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases} \quad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$



Properties of the Probability Density Function

- The numerical value f(x) is not a probability. It can be larger than 1.
- f(x)dx can be intepreted as the probability $P(x < X \le x + dx)$ since

$$P(x < X \le x + dx) = F(x + dx) - F(x) \approx f(x) dx$$

- $P(a \le X \le b) = \int_a^b f(x) dx$
- $\int_{-\infty}^{\infty} f(x) dx = 1$
- P(X = x) = 0 for all $x \in \mathbb{R}$

Independence

- Continuous random variables X and Y are independent if the events {X ≤ x} and {Y ≤ y} are independent for all x and y in ℝ
- If X and Y are independent, then the random variables g(X) and h(Y) are independent
- Exercise
 - Let X and Y be independent continuous random variables with common distribution function F and density function f. Find the density functions of max(X, Y) and min(X, Y).

Jointly Distributed Continuous Random Variables

Jointly Distributed Continuous Random Variables

Definition

The joint probability distribution function of RVs X and Y is given by

$$F_{X,Y}(x,y) = P\left(X \leq x \bigcap Y \leq y\right) = P(X \leq x, Y \leq y).$$

X and Y are said to be jointly continuous random variables with joint pdf $f_{X,Y}(x, y)$ if

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v) \, du \, dv$$

for all x, y in \mathbb{R}

Definition

Given the joint pdf, the marginal pdfs are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx$$

Properties of the Joint PDF

•
$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \, dy = 1$$

• X and Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$
 for all $x, y \in \mathbb{R}$

Exercise

- The joint probability density function of two continuous random variables X and Y is given by f(x, y) = c(2x + y) where x and y take real values such that 0 ≤ x ≤ 2, 0 ≤ y ≤ 3, and f(x, y) = 0 otherwise. Find the value of c.
- Given independent random variables *X*₁, *X*₂, ..., *X_n* with probability density functions *f*₁, *f*₂, ..., *f_n* respectively, find the probability density functions of the following
 - $\max(X_1, X_2, ..., X_n)$
 - $\min(X_1, X_2, \ldots, X_n)$

Conditional Distribution Function

- For discrete RVs, the conditional distribution was defined as $F_{Y|X}(y|x) = P(Y \le y|X = x)$ for any x such that P(X = x) > 0
- For continuous RVs, P(X = x) = 0 for all x
- But considering an interval around *x* such that *f_X(x) >* 0, we have

$$P(Y \le y | x \le X \le x + dx) = \frac{P(Y \le y, x \le X \le x + dx)}{P(x \le X \le x + dx)}$$
$$\approx \frac{\int_{v = -\infty}^{y} f(x, v) \, dx \, dv}{f_X(x) \, dx}$$
$$= \int_{v = -\infty}^{y} \frac{f(x, v)}{f_X(x)} \, dv$$

Definition

The conditional distribution function of *Y* given X = x is the function $F_{Y|X}(\cdot|x)$ given by

$$F_{Y|X}(y|x) = \int_{v=-\infty}^{y} \frac{f(x,v)}{f_X(x)} dv$$

for any x such that $f_X(x) > 0$. It is sometimes denoted by $P(Y \le y | X = x)$.

Conditional Density Function

Definition

The conditional density function of *Y* given X = x is given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

for any x such that $f_X(x) > 0$.

Properties

•
$$\int_{-\infty}^{\infty} f_{Y|X}(y|x) dy = 1$$

•
$$\int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) \, dx = f_Y(y)$$

Sum of Continuous Random Variables

Theorem

If X and Y have a joint density function f, then X + Y has density function

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x) \, dx.$$

If X and Y are independent, then

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) \, dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) \, dy.$$

The density function of the sum is the convolution of the marginal density functions.

Example (Sum of Uniform RVs)

Let $X \sim U[0, 1]$ and $Y \sim U[0, 1]$ be independent. What is the density function of X + Y?

Reading Assignment

 Sections 2.1, 2.3, 2.4, 2.5, 3.1, 3.2, 4.1, 4.2 from *Probability and Random Processes*, G. Grimmett and D. R. Stirzaker, 2001 (3rd Edition)