Why is the Probability Space a Triple?

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Probability Space

Definition

A probability space is a triple $(\Omega, \mathcal{F}, P)$ consisting of

- a set Ω,
- a σ-field $\mathcal{F}$ of subsets of Ω and
- a probability measure P on $(\Omega, \mathcal{F})$.

Remarks

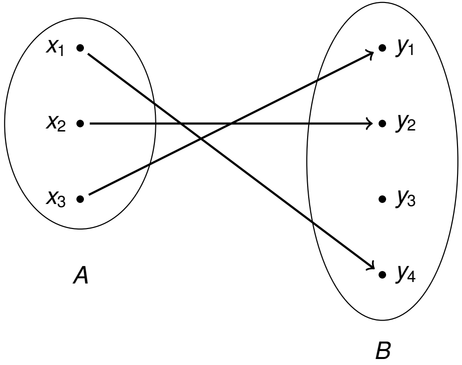
- When Ω is finite or countable, $\mathcal{F}$ can be 2^Ω (all subsets can be events)
- If this always holds, then Ω uniquely specifies $\mathcal{F}$
- Then the probability space would be an ordered pair (Ω, P)
- For uncountable Ω, it may be impossible to define P if $\mathcal{F} = 2^\Omega$
- We will see an example but first we need the following definitions
  - Countable and uncountable sets
  - Equivalence relations

Countable and Uncountable Sets

One-to-One Functions

Definition (One-to-One function)

A function $f : A \to B$ is a one-to-one function if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ and $x_1, x_2 \in A$.

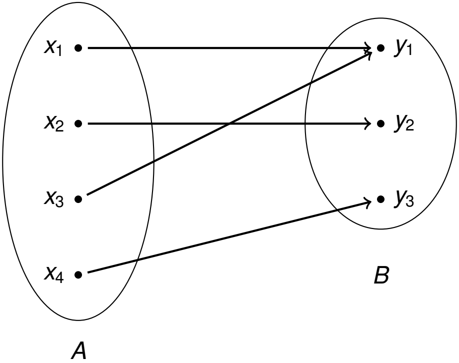


Also called an injective function

Onto Functions

Definition (Onto function)

A function $f : A \to B$ is said to be an onto function if $f(A) = B$.

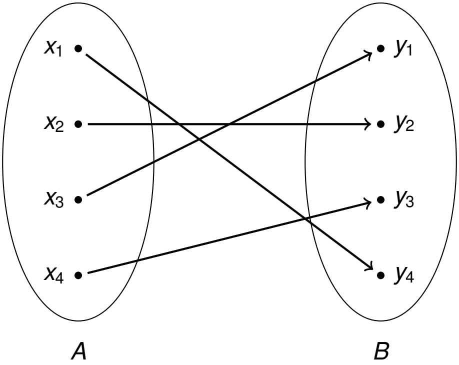


Also called a surjective function

One-to-One Correspondence

Definition (One-to-one correspondence)

A function $f : A \to B$ is said to be a one-to-one correspondence if it is a one-to-one and onto function from A to B.



Also called a bijective function

Countable Sets

Definition

Sets A and B are said to have the same cardinality if there exists a one-to-one correspondence $f : A \to B$.

Definition (Countable Sets)

A set A is said to be countable if there exists a one-to-one correspondence between A and $\mathbb{N}$.

Examples

- $\mathbb{N}$ is countable. Consider $f : \mathbb{N} \to \mathbb{N}$ defined as

$$f(x) = x$$

- $\mathbb{Z}$ is countable. Consider $f : \mathbb{Z} \to \mathbb{N}$ defined as

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$$

More Examples of Countable Sets

$$\begin{tikzcd}
(1,1) \arrow[r] & (2,1) \arrow[dl] & (3,1) \arrow[r] & (4,1) \arrow[dl] & (5,1) & \cdots \\
(1,2) \arrow[d] & (2,2) \arrow[ur] & (3,2) \arrow[dl] & (4,2) \arrow[ur] & (5,2) & \cdots \\
(1,3) \arrow[ur] & (2,3) \arrow[dl] & (3,3) \arrow[ur] & (4,3) & (5,3) & \cdots \\
(1,4) \arrow[d] & (2,4) \arrow[ur] & (3,4) & (4,4) & (5,4) & \cdots \\
(1,5) \arrow[ur] & (2,5) & (3,5) & (4,5) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{tikzcd}$$

- Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $f(i, j)$ is equal to the number of pairs visited when (i, j) is visited
- $\mathbb{N} \times \mathbb{N}$ is countable
- The same argument applies to any $A \times B$ where A and B are countable
- $\mathbb{Z} \times \mathbb{N}$ is countable $\implies$ $\mathbb{Q}$ is countable

Reals are Uncountable

Definition (Uncountable Sets)

A set is said to be uncountable if it is neither finite nor countable.

Examples

- $[0, 1)$ is uncountable
- $\mathbb{R}$ is uncountable

Equivalence Relations

Binary Relations

Definition (Binary Relation)

Given a set A, a binary relation R is a subset of $A \times A$.

Examples

- $A = \{1, 2, 3, 4\}, R = \{(1, 1), (2, 4)\}$
- $R = \left\{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \middle| a - b \text{ is an even integer} \right\}$
- $R = \left\{ (X, Y) \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} \middle| \text{ A bijection exists between } X \text{ and } Y \right\}$

If $(a, b) \in R$, we write $a \sim_R b$ or just $a \sim b$.

Equivalence Relations

Definition (Equivalence Relation)

A binary relation R on a set A is said to be an equivalence relation on A if for all $x, y, z \in A$ the following conditions hold

Reflexive	$x \sim x$
Symmetric	$x \sim y$ implies $y \sim x$
Transitive	$x \sim y$ and $y \sim z$ imply $x \sim z$

Examples

- $A = \{1, 2, 3, 4\}, R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
- $R = \left\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \middle| x - y \text{ is an even integer} \right\}$
- $R = \left\{ (x, y) \in \mathbb{Z} \times \mathbb{Z} \middle| x - y \text{ is a multiple of } 5 \right\}$
- Let A be the set of current students in the institute. Are the following binary relations equivalence relations on A?
    - $x \sim y$ if x and y live in the same hostel
    - $x \sim y$ if x and y have a course in common

Equivalence Classes

Definition (Equivalence Class)

Given an equivalence relation R on A and an element $x \in A$, the equivalence class of x is the set of all $y \in A$ such that $x \sim y$.

Examples

- $A = \{1, 2, 3, 4\}, R = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$
  Equivalence class of 1 is $\{1\}$.
- $R = \left\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \middle| a - b \text{ is an even integer}\right\}$
  Equivalence class of 0 is the set of all even integers.
  Equivalence class of 1 is the set of all odd integers.
- $R = \left\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \middle| a - b \text{ is a multiple of } 5\right\}$. Equivalence classes?

Theorem

Given an equivalence relation, the collection of equivalence classes form a partition of A.

A Non-Measurable Set

Choosing a Random Point in the Unit Interval

- Let $\Omega = [0, 1]$
- For $0 \leq a \leq b \leq 1$, we want

$$P([a, b]) = P((a, b]) = P([a, b)) = P((a, b)) = b - a$$

- We want P to be unaffected by shifting (with wrap-around)

$$P\left([0, 0.5]\right) = P\left([0.25, 0.75]\right) = P\left([0.75, 1] \cup [0, 0.25]\right)$$

- In general, for each subset $A \subseteq [0, 1]$ and $0 \leq r \leq 1$

$$P(A \oplus r) = P(A)$$

  where $\oplus$ indicates a circular shift in $[0, 1]$, i.e.

$$A \oplus r = \{a + r | a \in A, a + r \leq 1\} \cup \{a + r - 1 | a \in A, a + r > 1\}$$

- We want P to be countably additive

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

  for disjoint subsets $A_1, A_2, \ldots$ of $[0, 1]$
- Can the definition of P be extended to all subsets of $[0, 1]$?

Building the Contradiction

- Suppose P is defined for all subsets of $[0, 1]$
- Define an equivalence relation on $[0, 1]$ given by

$$x \sim y \iff x - y \text{ is rational}$$

- This relation partitions $[0, 1]$ into disjoint equivalence classes
- Let H be a subset of $[0, 1]$ consisting of exactly one element from each equivalence class. Let $0 \in H$; then $1 \notin H$.
- $[0, 1)$ is equal to $\bigcup_{r \in [0,1) \cap \mathbb{Q}} (H \oplus r)$
- Since the sets $H \oplus r$ for $r \in [0, 1) \cap \mathbb{Q}$ are disjoint, by countable additivity

$$P([0, 1)) = \sum_{r \in [0,1) \cap \mathbb{Q}} P(H \oplus r)$$

- Shift invariance implies $P(H \oplus r) = P(H)$ which implies

$$1 = P([0, 1)) = \sum_{r \in [0,1) \cap \mathbb{Q}} P(H)$$

which is a contradiction

Consequences of the Contradiction

- P cannot be defined on all subsets of $[0, 1]$
- But the subsets it is defined on have to form a σ-field
- The σ-field of subsets of $[0, 1]$ on which P can be defined without contradiction are called the measurable subsets
- That is why probability spaces are triples

Reference

- Chapter 1 from *A First Look at Rigorous Probability Theory*, Jeffrey S. Rosenthal, 2006 (2nd Edition)