EE 605: Error Correcting Codes (Autumn 2015)
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1. [5 points] Determine which of the following sets are groups under addition. A nonzero rational number is said to be in lowest terms when the numerator and denominator have a gcd of either 1 or -1.
   (a) The set of rational numbers (including 0) in lowest terms whose denominators are odd.
   (b) The set of rational numbers (including 0) in lowest terms whose denominators are even.
   (c) The set of rational numbers of absolute value less than 1.
   (d) The set of rational numbers of absolute value greater than or equal to 1 together with zero.
2. [5 points] Let $G = \{z \in \mathbb{C} | z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$.
   (a) Prove that G is a group under multiplication.
   (b) Prove that G is not a group under addition.
3. [5 points] Let $G = \{a + b\sqrt{2} \in \mathbb{R} | a, b \in \mathbb{Q}\}$.
   (a) Prove that G is a group under addition.
   (b) Prove that the nonzero elements of G form a group under multiplication.
4. [5 points] Prove that the inverse element of an element g in a group G is unique.
5. [5 points] Prove that the identity element in a group G is unique.
6. [5 points] Prove that $A \times B$ is an abelian group if and only if both A and B are abelian groups.
7. [5 points] Let m be a positive integer. If m is not a prime, prove that the set $\{1, 2, 3, \ldots, m-1\}$ is not a group under modulo-m multiplication.
8. [5 points] Prove that a subset H of a group G is a subgroup if it is nonempty, finite and closed under the group operation. *Hint: Use finiteness to show that every element has an inverse.*
9. [5 points] Give an example of a group G and an infinite subset H of G that is closed under the group operation but is not a subgroup of G.
10. [5 points] Let H and K be subgroups of a group G. Prove that $H \cup K$ is a subgroup of G if and only if H is a subgroup of K or K is a subgroup of H.
11. [5 points] Prove that if H and K are subgroups of a group G then so is their intersection $H \cap K$.
12. [5 points] Prove that G cannot have a subgroup H with $|H| = n - 1$, where $n = |G| > 2$.
13. [5 points] If $\phi : G \to H$ is an isomorphism between groups G and H, show that $\phi(0_G) = 0_H$ where 0_G is the additive identity of G and 0_H is the additive identity of H