# Hypothesis Testing

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# Basics of Hypothesis Testing

# What is a Hypothesis?

One situation among a set of possible situations

**Example (Radar)**

EM waves are transmitted and the reflections observed.

**Null Hypothesis** Plane absent

**Alternative Hypothesis** Plane present

For a given set of observations, either hypothesis may be true.

# What is Hypothesis Testing?

- A statistical framework for deciding which hypothesis is true
- Under each hypothesis the observations are assumed to have a known distribution
- Consider the case of two hypotheses (binary hypothesis testing)

$$\begin{aligned} H_0 &: \mathbf{Y} \sim P_0 \\ H_1 &: \mathbf{Y} \sim P_1 \end{aligned}$$

  $\mathbf{Y}$ is the random observation vector belonging to observation set $\Gamma \subseteq \mathbb{R}^n$ for $n \in \mathbb{N}$
- The hypotheses are assumed to occur with given prior probabilities

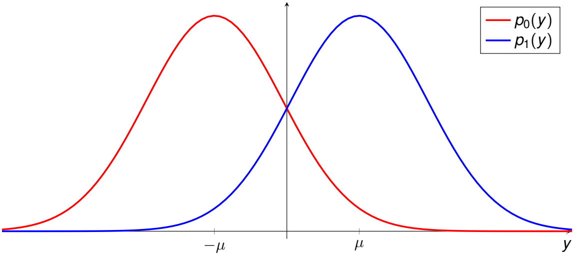
$$\begin{aligned} \Pr(H_0 \text{ is true}) &= \pi_0 \\ \Pr(H_1 \text{ is true}) &= \pi_1 \end{aligned}$$

  where $\pi_0 + \pi_1 = 1$.

# Location Testing with Gaussian Error

- Let observation set $\Gamma = \mathbb{R}$ and $\mu > 0$

$$\begin{aligned} H_0 &: Y \sim N(-\mu, \sigma^2) \\ H_1 &: Y \sim N(\mu, \sigma^2) \end{aligned}$$




- Any point in $\Gamma$ can be generated under both $H_0$ and $H_1$
- What is a good decision rule for this hypothesis testing problem which takes the prior probabilities into account?

# What is a Decision Rule?

- A decision rule for binary hypothesis testing is a partition of $\Gamma$ into $\Gamma_0$ and $\Gamma_1$ such that

$$\delta(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{y} \in \Gamma_0 \\ 1 & \text{if } \mathbf{y} \in \Gamma_1 \end{cases}$$

  We decide $H_i$ is true when $\delta(\mathbf{y}) = i$ for $i \in \{0, 1\}$
- For the location testing with Gaussian error problem, one possible decision rule is

$$\begin{aligned} \Gamma_0 &= (-\infty, 0] \\ \Gamma_1 &= (0, \infty) \end{aligned}$$

  and another possible decision rule is

$$\begin{aligned} \Gamma_0 &= (-\infty, -100) \cup (-50, 0) \\ \Gamma_1 &= [-100, -50] \cup [0, \infty) \end{aligned}$$

- Given that partitions of the observation set define decision rules, what is the optimal partition?

# Which is the Optimal Decision Rule?

- Minimizing the probability of decision error gives the optimal decision rule
- For the binary hypothesis testing problem of $H_0$ versus $H_1$, the conditional decision error probability given $H_i$ is true is

$$\begin{aligned} P_{e|i} &= \Pr\left[\text{Deciding } H_{1-i} \text{ is true} | H_i \text{ is true}\right] \\ &= \Pr\left[Y \in \Gamma_{1-i} | H_i\right] \\ &= 1 - \Pr\left[Y \in \Gamma_i | H_i\right] \\ &= 1 - P_{c|i} \end{aligned}$$

- Probability of decision error is

$$P_e = \pi_0 P_{e|0} + \pi_1 P_{e|1}$$

- Probability of correct decision is

$$P_c = \pi_0 P_{c|0} + \pi_1 P_{c|1} = 1 - P_e$$

# Which is the Optimal Decision Rule?

- Maximizing the probability of correct decision will minimize probability of decision error
- Probability of correct decision is

$$\begin{aligned} P_c &= \pi_0 P_{c|0} + \pi_1 P_{c|1} \\ &= \pi_0 \int_{y \in \Gamma_0} p_0(y)\, dy + \pi_1 \int_{y \in \Gamma_1} p_1(y)\, dy \end{aligned}$$

- If a point $y$ in $\Gamma$ belongs to $\Gamma_i$, its contribution to $P_c$ is proportional to $\pi_i p_i(y)$
- To maximize $P_c$, we choose the partition $\{\Gamma_0, \Gamma_1\}$ as

$$\begin{aligned} \Gamma_0 &= \{y \in \Gamma | \pi_0 p_0(y) \geq \pi_1 p_1(y)\} \\ \Gamma_1 &= \{y \in \Gamma | \pi_1 p_1(y) > \pi_0 p_0(y)\} \end{aligned}$$

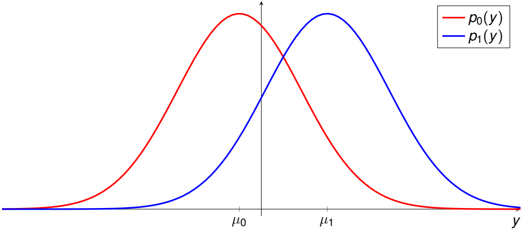
- The points $y$ for which $\pi_0 p_0(y) = \pi_1 p_1(y)$ can be in either $\Gamma_0$ and $\Gamma_1$ (the optimal decision rule is not unique)

# Location Testing with Gaussian Error

- Let $\mu_1 > \mu_0$ and $\pi_0 = \pi_1 = \frac{1}{2}$

$$\begin{aligned} H_0 &: \quad Y = \mu_0 + Z \\ H_1 &: \quad Y = \mu_1 + Z \end{aligned}$$

where $Z \sim N(0, \sigma^2)$




$$\begin{aligned} p_0(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_0)^2}{2\sigma^2}} \\ p_1(y) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu_1)^2}{2\sigma^2}} \end{aligned}$$

# Location Testing with Gaussian Error

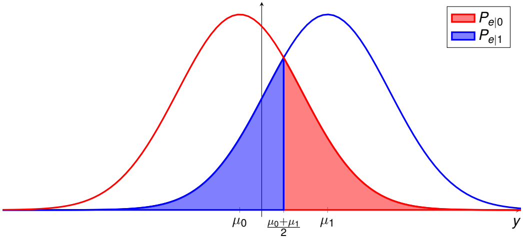
- Optimal decision rule is given by the partition $\{\Gamma_0, \Gamma_1\}$

$$
\begin{aligned}
\Gamma_0 &= \{y \in \Gamma | \pi_0 p_0(y) \geq \pi_1 p_1(y)\} \\
\Gamma_1 &= \{y \in \Gamma | \pi_1 p_1(y) > \pi_0 p_0(y)\}
\end{aligned}
$$

- For $\pi_0 = \pi_1 = \frac{1}{2}$

$$
\begin{aligned}
\Gamma_0 &= \left\{ y \in \Gamma \middle| y \leq \frac{\mu_1 + \mu_0}{2} \right\} \\
\Gamma_1 &= \left\{ y \in \Gamma \middle| y > \frac{\mu_1 + \mu_0}{2} \right\}
\end{aligned}
$$

# Location Testing with Gaussian Error




$$P_{e|0} = \Pr\left[ Y > \frac{\mu_0 + \mu_1}{2} \middle| H_0 \right] = Q\left( \frac{\mu_1 - \mu_0}{2\sigma} \right)$$

$$P_{e|1} = \Pr\left[ Y \leq \frac{\mu_0 + \mu_1}{2} \middle| H_1 \right] = \Phi\left( \frac{\mu_0 - \mu_1}{2\sigma} \right) = Q\left( \frac{\mu_1 - \mu_0}{2\sigma} \right)$$

$$P_e = \pi_0 P_{e|0} + \pi_1 P_{e|1} = Q\left( \frac{\mu_1 - \mu_0}{2\sigma} \right)$$

This $P_e$ is for $\pi_0 = \pi_1 = \frac{1}{2}$

# Location Testing with Gaussian Error

- Suppose $\pi_0 \neq \pi_1$
- Optimal decision rule is still given by the partition $\{\Gamma_0, \Gamma_1\}$

$$
\begin{aligned}
\Gamma_0 &= \{y \in \Gamma | \pi_0 p_0(y) \geq \pi_1 p_1(y)\} \\
\Gamma_1 &= \{y \in \Gamma | \pi_1 p_1(y) > \pi_0 p_0(y)\}
\end{aligned}
$$

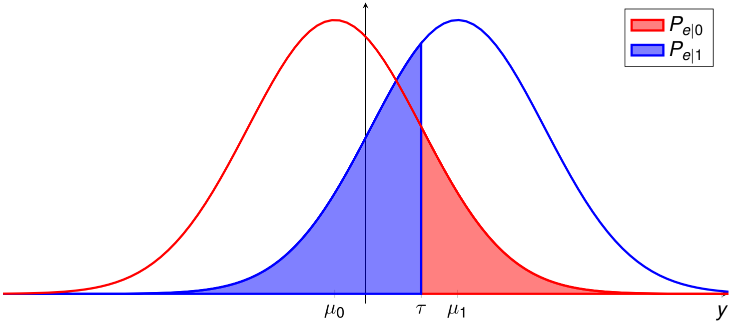
- The partitions specialized to this problem are

$$
\begin{aligned}
\Gamma_0 &= \left\{ y \in \Gamma \middle| y \leq \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} \right\} \\
\Gamma_1 &= \left\{ y \in \Gamma \middle| y > \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} \right\}
\end{aligned}
$$

# Location Testing with Gaussian Error

Suppose $\pi_0 = 0.6$ and $\pi_1 = 0.4$

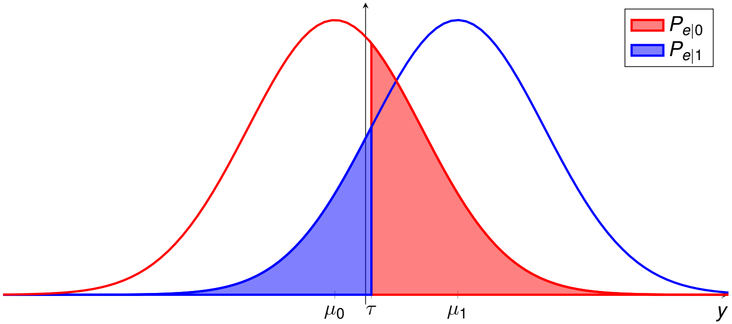
$$\tau = \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} = \frac{\mu_1 + \mu_0}{2} + \frac{0.4054\sigma^2}{(\mu_1 - \mu_0)}$$



# Location Testing with Gaussian Error

Suppose $\pi_0 = 0.4$ and $\pi_1 = 0.6$

$$\tau = \frac{\mu_1 + \mu_0}{2} + \frac{\sigma^2}{(\mu_1 - \mu_0)} \log \frac{\pi_0}{\pi_1} = \frac{\mu_1 + \mu_0}{2} - \frac{0.4054\sigma^2}{(\mu_1 - \mu_0)}$$



# $M$-ary Hypothesis Testing

- $M$ hypotheses with prior probabilities $\pi_i$, $i = 1, \ldots, M$

$$
\begin{array}{ccc}
H_1 & : & \mathbf{Y} \sim P_1 \\
H_2 & : & \mathbf{Y} \sim P_2 \\
\vdots & & \vdots \\
H_M & : & \mathbf{Y} \sim P_M
\end{array}
$$

- A decision rule for $M$-ary hypothesis testing is a partition of $\Gamma$ into $M$ disjoint regions $\{\Gamma_i | i = 1, \ldots, M\}$ such that

$$\delta(\mathbf{y}) = i \text{ if } \mathbf{y} \in \Gamma_i$$

  We decide $H_i$ is true when $\delta(\mathbf{y}) = i$ for $i \in \{1, \ldots, M\}$

- Minimum probability of error rule is

$$\delta_{\mathrm{MPE}}(\mathbf{y}) = \arg \max_{1 \le i \le M} \pi_i p_i(\mathbf{y})$$

# Maximum A Posteriori Decision Rule

- The a posteriori probability of $H_i$ being true given observation $\mathbf{y}$ is

$$P\left[H_i \text{ is true} \middle| \mathbf{y}\right] = \frac{\pi_i p_i(\mathbf{y})}{p(\mathbf{y})}$$

- The MAP decision rule is given by

$$\delta_{\text{MAP}}(\mathbf{y}) = \arg \max_{1 \le i \le M} P\left[H_i \text{ is true} \middle| \mathbf{y}\right] = \delta_{\text{MPE}}(\mathbf{y})$$

MAP decision rule = MPE decision rule

# Maximum Likelihood Decision Rule

- The ML decision rule is given by

$$\delta_{\text{ML}}(\mathbf{y}) = \arg \max_{1 \le i \le M} p_i(\mathbf{y})$$

- If the $M$ hypotheses are equally likely, $\pi_i = \frac{1}{M}$
- The MPE decision rule is then given by

$$\delta_{\text{MPE}}(\mathbf{y}) = \arg \max_{1 \le i \le M} \pi_i p_i(\mathbf{y}) = \delta_{\text{ML}}(\mathbf{y})$$

For equal priors, ML decision rule = MPE decision rule

# Irrelevant Statistics

# Irrelevant Statistics

- In this context, the term statistic means an observation
- For a given hypothesis testing problem, all the observations may not be useful

## Example (Irrelevant Statistic)

$$\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$$

$$\begin{aligned} H_1 &: \quad Y_1 = A + N_1, \quad Y_2 = N_2 \\ H_0 &: \quad Y_1 = N_1, \qquad\quad\; Y_2 = N_2 \end{aligned}$$

where $A > 0$, $N_1 \sim N(0, \sigma^2)$, $N_2 \sim N(0, \sigma^2)$.

- If $N_1$ and $N_2$ are independent, $Y_2$ is irrelevant.
- If $N_1$ and $N_2$ are correlated, $Y_2$ is relevant.

- Need a method to recognize irrelevant components of the observations

# Characterizing an Irrelevant Statistic

## Theorem

*For $M$-ary hypothesis testing using an observation $\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{Y}_2 \end{bmatrix}$, the statistic $\mathbf{Y}_2$ is irrelevant if the conditional distribution of $\mathbf{Y}_2$, given $\mathbf{Y}_1$ and $H_i$, is independent of $i$. In terms of densities, the condition for irrelevance is*

$$p(\mathbf{y}_2|\mathbf{y}_1, H_i) = p(\mathbf{y}_2|\mathbf{y}_1) \ \ \forall i.$$

## Proof

$$\begin{aligned}
\delta_{\text{MPE}}(\mathbf{y}) &= \arg \max_{1 \le i \le M} \pi_i p_i(\mathbf{y}) = \arg \max_{1 \le i \le M} \pi_i p(\mathbf{y}|H_i) \\
p(\mathbf{y}|H_i) &= p(\mathbf{y}_1, \mathbf{y}_2|H_i) = p(\mathbf{y}_2|\mathbf{y}_1, H_i) p(\mathbf{y}_1|H_i) \\
&= p(\mathbf{y}_2|\mathbf{y}_1) p(\mathbf{y}_1|H_i)
\end{aligned}$$

# Example of an Irrelevant Statistic

## Example (Independent Noise)

$$\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$$

$$\begin{array}{ll} H_1: & Y_1 = A + N_1, \quad Y_2 = N_2 \\ H_0: & Y_1 = N_1, \qquad\quad\; Y_2 = N_2 \end{array}$$

where $A > 0$, $N_1 \sim N(0, \sigma^2)$, $N_2 \sim N(0, \sigma^2)$, $N_1 \perp N_2$.

$$\begin{aligned} p(\mathbf{y}_2 | \mathbf{y}_1, H_0) &= p(\mathbf{y}_2) \\ p(\mathbf{y}_2 | \mathbf{y}_1, H_1) &= p(\mathbf{y}_2) \end{aligned}$$

# Example of a Relevant Statistic

## Example (Correlated Noise)

$$\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}^T$$

$$\begin{array}{lll} H_1: & Y_1 = A + N_1, & Y_2 = N_2 \\ H_0: & Y_1 = N_1, & Y_2 = N_2 \end{array}$$

where $A > 0$, $N_1 \sim N(0, \sigma^2)$, $N_2 \sim N(0, \sigma^2)$, $\mathbf{C}_Y = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

$$\begin{aligned} p(y_2|y_1, H_0) &= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma^2}} e^{-\frac{(y_2-\rho y_1)^2}{2(1-\rho^2)\sigma^2}}, \\ p(y_2|y_1, H_1) &= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma^2}} e^{-\frac{[y_2-\rho(y_1-A)]^2}{2(1-\rho^2)\sigma^2}} \end{aligned}$$

Thanks for your attention