ML Performance of *M*-ary Signaling

Saravanan Vijayakumaran
sarva@ee.iitb.ac.in

Department of Electrical Engineering
Indian Institute of Technology Bombay

October 10, 2012

Performance of ML Decision Rule for M-ary signaling

ML Decision Rule for M-ary Signaling

The ML decision rule for M-ary signaling in a real AWGN channel is

$$\delta_{ML}(\mathbf{y}) = \arg \min_{1 \le i \le M} \|\mathbf{y} - \mathbf{s}_i\|^2 = \arg \max_{1 \le i \le M} \left[\langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2} \right]$$

The ML decision rule for M-ary signaling in a complex AWGN channel is

$$\delta_{ML}(\mathbf{y}) = \arg \min_{1 \le i \le M} \|\mathbf{y} - \mathbf{s}_i\|^2 = \arg \max_{1 \le i \le M} \left[\operatorname{Re}\left(\langle \mathbf{y}, \mathbf{s}_i \rangle \right) - \frac{\|\mathbf{s}_i\|^2}{2} \right]$$

In both cases, the rule can be represented as

$$\delta_{ML}(\mathbf{y}) = \arg \max_{1 \le i \le M} Z_i$$

where Z_i is the decision statistic

ML Decision Rule for Binary Signaling

ML decision rule

$$\delta_{ML}(\mathbf{y}) = \arg\max_{1 \le i \le 2} Z_i = \arg\max_{1 \le i \le 2} \left[\langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2} \right]$$

Probability of error

$$P_e = Q\left(\frac{\|s_0 - s_1\|}{2\sigma} \right) = Q\left(\sqrt{\frac{\|s_0 - s_1\|^2}{2N_0}} \right)$$

Let $E_b = \frac{1}{2}\left(\|s_0\|^2 + \|s_1\|^2\right)$. For antipodal signaling,

$$P_e = Q\left(\sqrt{\frac{2E_b}{N_0}} \right)$$

ML Decision Rule for Binary Signaling

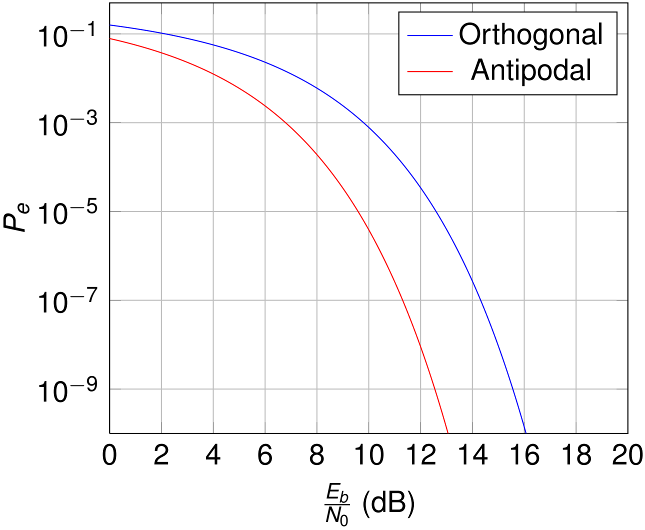
For on-off keying, $s_1(t) = s(t)$ and $s_0(t) = 0$ and

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

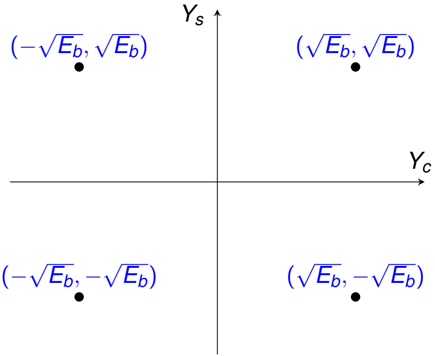
For orthogonal signaling, $s_1(t)$ and $s_2(t)$ are orthogonal

$$P_e = Q\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Performance Comparison of Antipodal and Orthogonal Signaling



ML Decision Rule for QPSK



$$P_{e|1} = \Pr\left[Y_c < 0 \text{ or } Y_s < 0 \middle| (\sqrt{E_b}, \sqrt{E_b}) \text{ was sent} \right]$$

ML Decision Rule for QPSK

$$\begin{aligned} P_{e|1} &= \Pr\left[Y_c < 0 \text{ or } Y_s < 0 \middle| (\sqrt{E_b}, \sqrt{E_b}) \text{ was sent} \right] \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{aligned}$$

By symmetry,

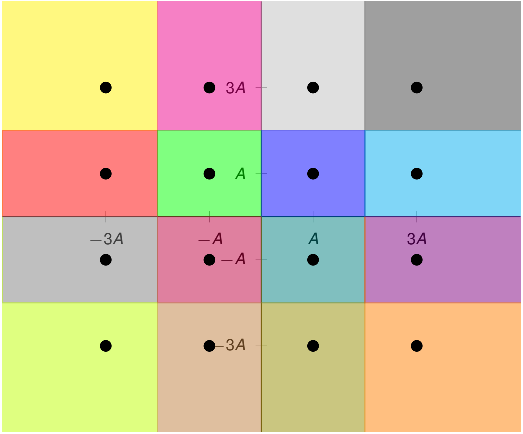
$$P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$$

Since the four constellation points are equally likely, the probability of error is given by

$$P_e = \frac{1}{4}\sum_{i=1}^{4} P_{e|i} = P_{e|1} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

ML Decision Rule for 16-QAM

16-QAM



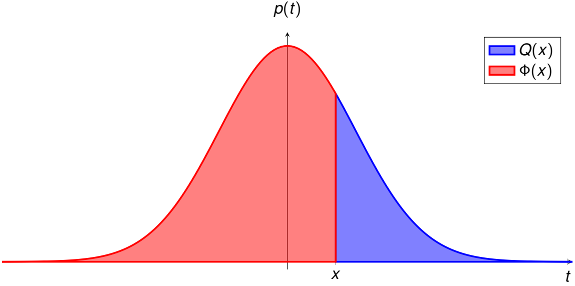
Exact analysis is tedious. Approximate analysis is sufficient.

Revisiting the Q function

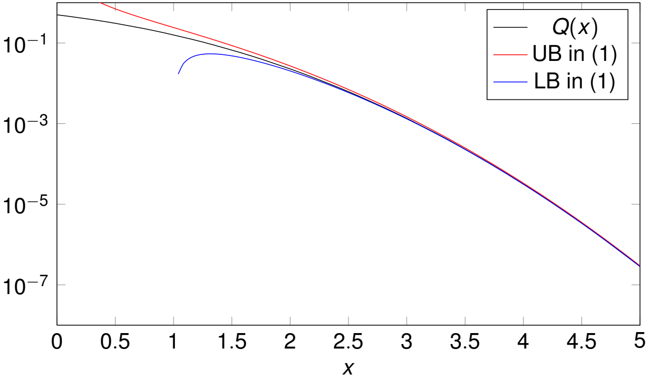
Revisiting the Q function

$X \sim N(0, 1)$

$$Q(x) = P\left[X > x\right] = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

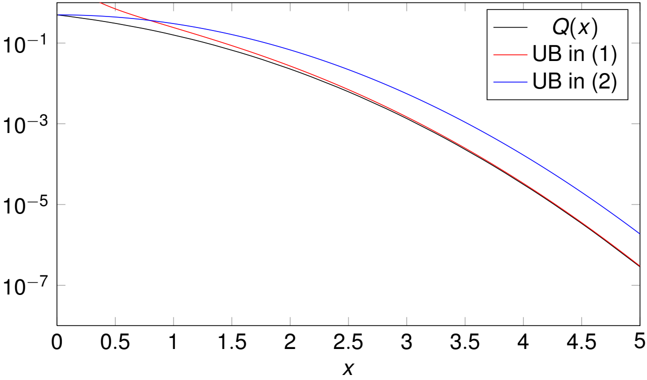


Bounds on $Q(x)$ for Large Arguments



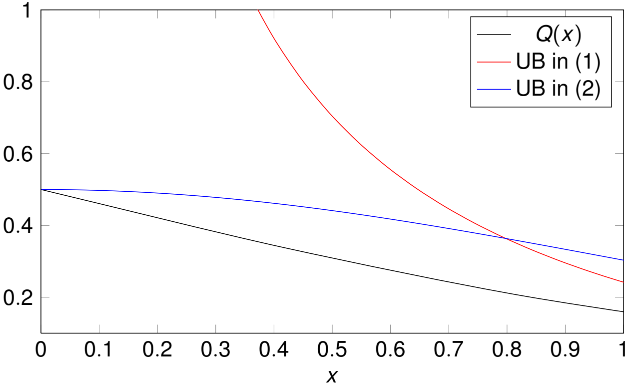
$$\left(1 - \frac{1}{x^2}\right) \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \tag{1}$$

Bounds on $Q(x)$ for Small Arguments

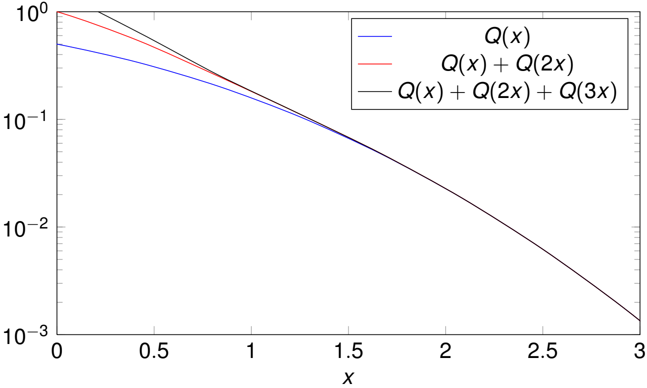


$$Q(x) \leq \frac{1}{2} e^{-\frac{x^2}{2}} \qquad (2)$$

Bounds on $Q(x)$ for Small Arguments



Q Functions with Smallest Arguments Dominate



- P_e in AWGN channels can typically be bounded by a sum of Q functions
- The Q function with the smallest argument is used to approximate P_e

Union Bound Analysis

Union Bound for M-ary Signaling in AWGN

The conditional error probability given H_i is true is

$$P_{e|i} = \Pr\left[\cup_{j \neq i}\{Z_i < Z_j\} \,\middle|\, H_i\right]$$

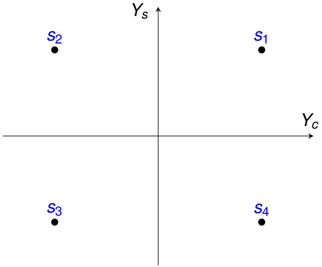
Since $P(A \cup B) \leq P(A) + P(B)$, we have

$$P_{e|i} \leq \sum_{j \neq i} \Pr\left[Z_i < Z_j \,\middle|\, H_i\right] = \sum_{j \neq i} Q\left(\frac{\|s_j - s_i\|}{2\sigma}\right)$$

The error probability for prior probabilities π_i is given by

$$P_e = \sum_i \pi_i P_{e|i} \leq \sum_i \pi_i \sum_{j \neq i} Q\left(\frac{\|s_j - s_i\|}{2\sigma}\right)$$

Union Bound for QPSK



$$
\begin{aligned}
P_{e|1} &= \Pr\left[\cup_{j\neq 1}\left\{Z_1 < Z_j\right\}\middle| H_1\right] \leq \sum_{j\neq 1}\Pr\left[Z_1 < Z_j \middle| H_1\right] \\
P_{e|1} &\leq Q\left(\frac{\|s_2 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_3 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_4 - s_1\|}{2\sigma}\right) \\
&= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)
\end{aligned}
$$

Union Bound for QPSK

Union bound on error probability of ML rule

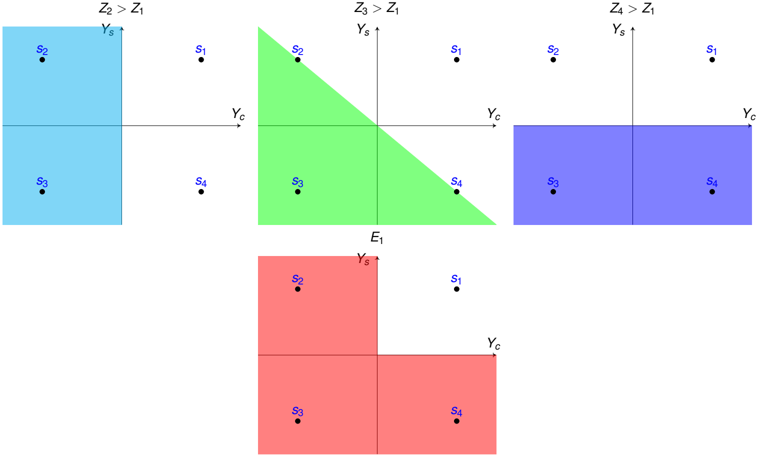
$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

Exact error probability of ML rule

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

QPSK Error Events

$$E_1 = [Z_2 > Z_1] \cup [Z_3 > Z_1] \cup [Z_4 > Z_1] = [Z_2 > Z_1] \cup [Z_4 > Z_1]$$



Intelligent Union Bound for QPSK

$$
\begin{aligned}
P_{e|1} &= \Pr\left[(Z_2 > Z_1) \cup (Z_4 > Z_1) \middle| H_1\right] \\
&\leq \Pr\left[Z_2 < Z_1 \middle| H_1\right] + \Pr\left[Z_2 < Z_1 \middle| H_1\right] \\
&= Q\left(\frac{\|s_2 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_4 - s_1\|}{2\sigma}\right) \\
&= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
\end{aligned}
$$

By symmetry $P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$ and

$$
P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
$$

Summary of results for QPSK

Exact error probability of ML rule

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

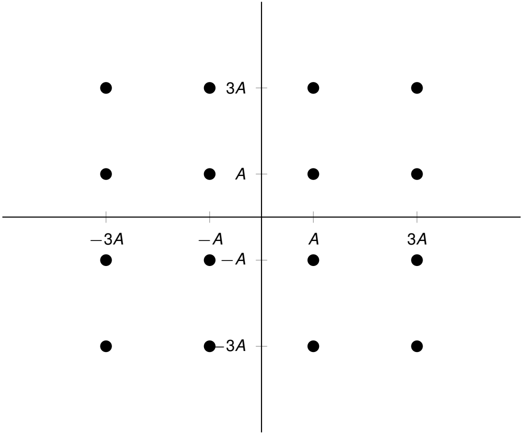
Union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

Intelligent union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

Intelligent Union Bound for 16-QAM



Assignment 4

Nearest Neighbors Approximation

Let d_{min} be the minimum distance between constellation points

$$d_{min} = \min_{i \neq j} \|s_i - s_j\|$$

Let $N_{d_{min}}(i)$ denote the number of nearest neighbors of s_i

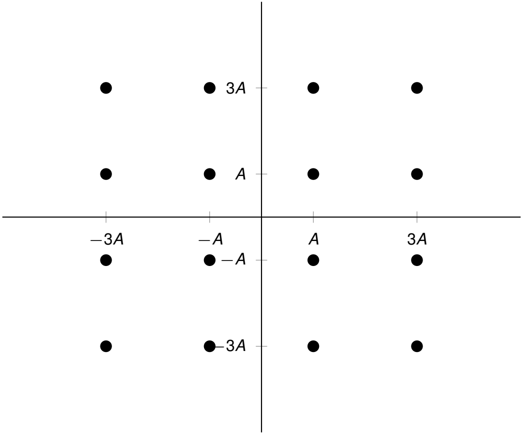
$$P_{e|i} \approx N_{d_{min}}(i) Q\left(\frac{d_{min}}{2\sigma}\right)$$

Averaging over i we get

$$P_e \approx \bar{N}_{d_{min}} Q\left(\frac{d_{min}}{2\sigma}\right)$$

where $\bar{N}_{d_{min}}$ denotes the average number of nearest neighbors

Nearest Neighbors Approximation for 16-QAM

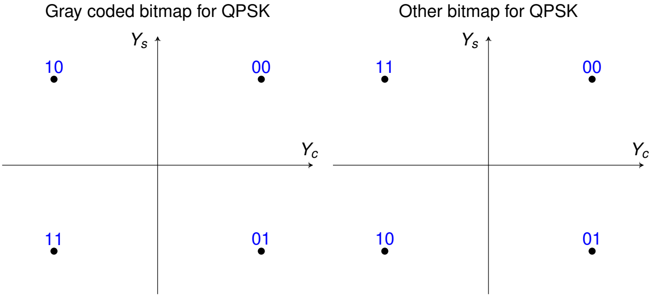


Assignment 4

Bit Error Probability of ML Rules

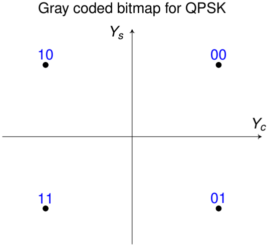
Bit Error Probability of ML Decision Rule

- Probability of bit error is also termed bit error rate (BER)
- For fixed SNR, symbol error probability depends only on constellation geometry
- For fixed SNR, BER depends on both constellation geometry and the bits to signal mapping



- For an M-ary constellation, number of possible bitmaps is $M! = M(M-1)\cdots 3\cdot 2\cdot 1$

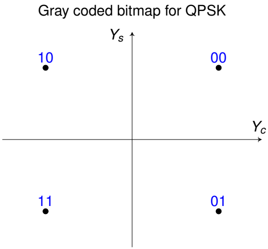
Bit Error Rate for QPSK using Gray Bitmap



Conditional BER when $b[1]b[2] = 00$ is

$$
\begin{aligned}
P_{b|00} &= \frac{1}{2} \Pr\left[\hat{b}[1]\hat{b}[2] = 01 \middle| b[1]b[2] = 00\right] \\
&+ \frac{1}{2} \Pr\left[\hat{b}[1]\hat{b}[2] = 10 \middle| b[1]b[2] = 00\right] \\
&+ \Pr\left[\hat{b}[1]\hat{b}[2] = 11 \middle| b[1]b[2] = 00\right]
\end{aligned}
$$

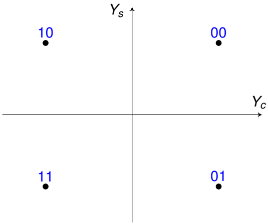
Bit Error Rate for QPSK using Gray Bitmap



Let $\alpha = \sqrt{\frac{2E_b}{N_0}}$

$$\begin{aligned}
\Pr\left[\hat{b}[1]\hat{b}[2] = 01 \middle| b[1]b[2] = 00\right] &= Q(\alpha)\left[1 - Q(\alpha)\right] \\
\Pr\left[\hat{b}[1]\hat{b}[2] = 10 \middle| b[1]b[2] = 00\right] &= Q(\alpha)\left[1 - Q(\alpha)\right] \\
\Pr\left[\hat{b}[1]\hat{b}[2] = 11 \middle| b[1]b[2] = 00\right] &= Q^2(\alpha)
\end{aligned}$$

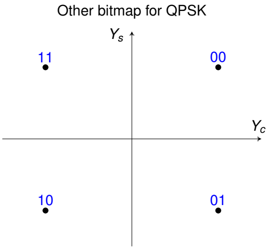
Bit Error Rate for QPSK using Gray Bitmap



Conditional BER when $b[1]b[2] = 00$ is

$$\begin{aligned} P_{b|00} &= \frac{1}{2}Q(\alpha)\left[1 - Q(\alpha)\right] + \frac{1}{2}Q(\alpha)\left[1 - Q(\alpha)\right] + Q^2(\alpha) \\ &= Q(\alpha) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \\ P_b &= \frac{1}{4}\left(P_{b|00} + P_{b|01} + P_{b|10} + P_{b|11}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{aligned}$$

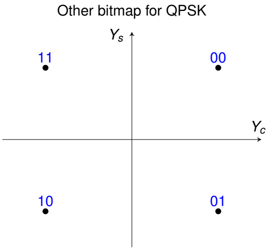
Bit Error Rate for QPSK using Other Bitmap



Conditional BER when $b[1]b[2] = 00$ is

$$
\begin{aligned}
P_{b|00} &= \frac{1}{2}\Pr\left[\hat{b}[1]\hat{b}[2] = 01 \middle| b[1]b[2] = 00\right] \\
&+ \frac{1}{2}\Pr\left[\hat{b}[1]\hat{b}[2] = 10 \middle| b[1]b[2] = 00\right] \\
&+ \Pr\left[\hat{b}[1]\hat{b}[2] = 11 \middle| b[1]b[2] = 00\right]
\end{aligned}
$$

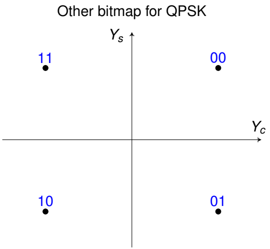
Bit Error Rate for QPSK using Other Bitmap



Let $\alpha = \sqrt{\frac{2E_b}{N_0}}$

$$
\begin{aligned}
\Pr\left[\hat{b}[1]\hat{b}[2] = 01 \middle| b[1]b[2] = 00\right] &= Q(\alpha)\left[1 - Q(\alpha)\right] \\
\Pr\left[\hat{b}[1]\hat{b}[2] = 10 \middle| b[1]b[2] = 00\right] &= Q^2(\alpha) \\
\Pr\left[\hat{b}[1]\hat{b}[2] = 11 \middle| b[1]b[2] = 00\right] &= Q(\alpha)\left[1 - Q(\alpha)\right]
\end{aligned}
$$

Bit Error Rate for QPSK using Other Bitmap

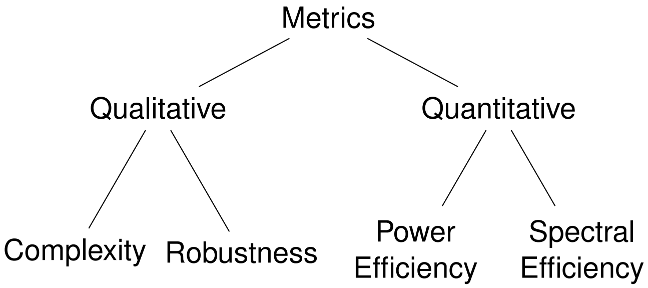


Conditional BER when $b[1]b[2] = 00$ is

$$
\begin{aligned}
P_{b|00} &= \frac{1}{2} Q(\alpha)\left[1 - Q(\alpha)\right] + \frac{1}{2} Q^2(\alpha) + Q(\alpha)\left[1 - Q(\alpha)\right] \\
&= \frac{3}{2} Q(\alpha) - Q^2(\alpha) \approx \frac{3}{2} Q(\alpha) = \frac{3}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \\
P_b &= \frac{1}{4}\left(P_{b|00} + P_{b|01} + P_{b|10} + P_{b|11}\right) \approx \frac{3}{2} Q\left(\sqrt{\frac{2E_b}{N_0}}\right)
\end{aligned}
$$

Comparison of Modulation Schemes

Metrics for Comparing Modulation Schemes



Power Efficiency

For an M-ary signaling scheme

$$
\begin{aligned}
P_e &\approx \bar{N}_{d_{min}} Q\left(\frac{d_{min}}{2\sigma}\right) \\
&= \bar{N}_{d_{min}} Q\left(\sqrt{\frac{d_{min}^2}{2N_0}}\right) = \bar{N}_{d_{min}} Q\left(\sqrt{\frac{d_{min}^2}{E_b}}\sqrt{\frac{E_b}{2N_0}}\right)
\end{aligned}
$$

The power efficiency of a modulation scheme is defined as

$$\eta_p = \frac{d_{min}^2}{E_b}$$

The nearest neighbors approximation can be expressed as

$$P_e \approx \bar{N}_{d_{min}} Q\left(\sqrt{\frac{\eta_p E_b}{2N_0}}\right)$$

Power Efficiency of Some Modulation Schemes

Modulation Scheme	η_p
On-off keying	2
Orthogonal signaling	2
Antipodal signaling	4
BPSK	4
QPSK	4
16-QAM	1.6

Spectral Efficiency

Definition (Spectral Efficiency)

The number of bits that can be transmitted using the modulation scheme per second per Hertz of bandwidth.

Remarks

- If a modulation scheme transmits N bits every T seconds using W Hertz of bandwidth, the spectral efficiency is $\frac{N}{WT}$ bits/s/Hz
- We will use null-to-null bandwidth to calculate spectral efficiency

Spectral Efficiency of BPSK

Let $S_p(f)$ be the PSD of BPSK and let $S(f)$ be the PSD of its complex envelope.

$$S_p(f) = \frac{S(f - f_c) + S(-f - f_c)}{2}$$

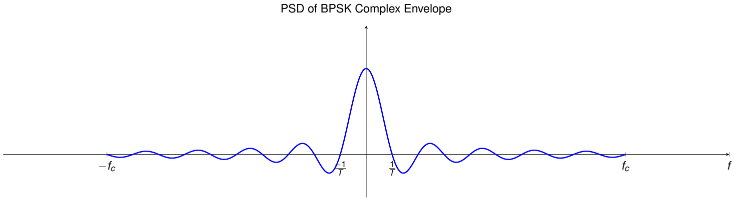
The complex envelope is given by

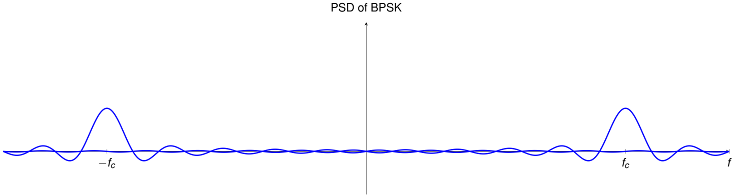
$$s(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

where $p(t)$ is a pulse of duration T and $b_n \in \{-A, A\}$.
Given $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$, PSD of the complex envelope is

$$S(f) = S_b\left(e^{j2\pi fT}\right) \frac{|P(f)|^2}{T} = A^2 T \operatorname{sinc}^2(fT)$$

Power Spectral Density of BPSK



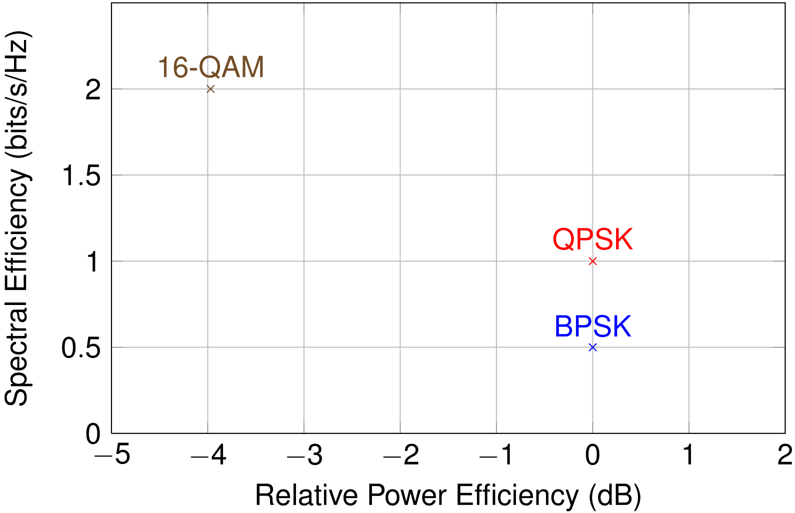


Null-to-null bandwidth of BPSK = $\frac{2}{T}$
Spectral Efficiency of BPSK = 0.5

Spectral Efficiency of Some Modulation Schemes

Modulation Scheme	Spectral Efficiency
BPSK	0.5
BPAM	1
QPSK	1
16-QAM	2

Spectral Efficiency vs Relative Power Efficiency



Thanks for your attention