Digital Modulation

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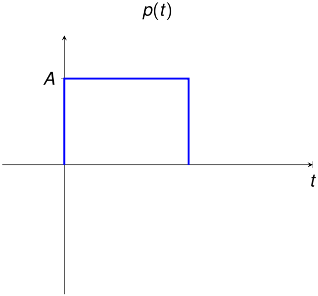
Digital Modulation

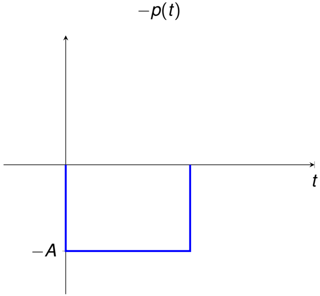
Definition

The process of mapping a bit sequence to signals for transmission over a channel.

Example (Binary Baseband PAM)

$1 \to p(t)$ and $0 \to -p(t)$





Classification of Modulation Schemes

- Memoryless
  - Divide bit sequence into k-bit blocks
  - Map each block to a signal $s_m(t), \; 1 \leq m \leq 2^k$
  - Mapping depends only on current k-bit block
- Having Memory
  - Mapping depends on current k-bit block and $L - 1$ previous blocks
  - L is called the constraint length
- Linear
  - Modulated signal has the form

$$u(t) = \sum_n b_n g(t - nT)$$

  where b_n's are the transmitted symbols and g is a fixed waveform
- Nonlinear

Signal Space Representation

Signal Space Representation of Waveforms

- Given M finite energy waveforms, construct an orthonormal basis

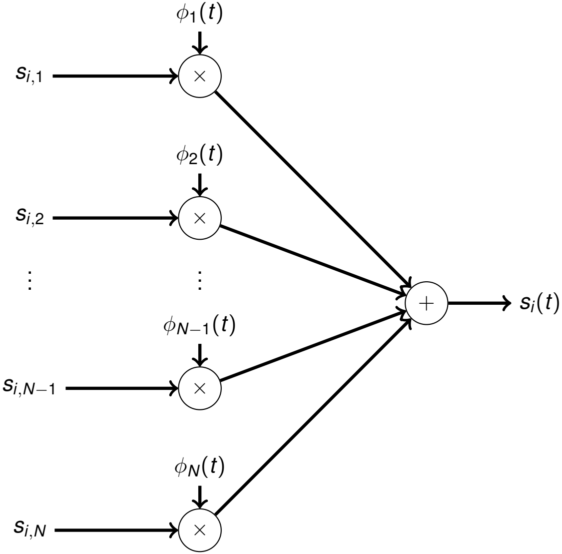
$$s_1(t), \ldots, s_M(t) \rightarrow \underbrace{\phi_1(t), \ldots, \phi_N(t)}_{\text{Orthonormal basis}}$$

- Each $s_i(t)$ is a linear combination of the basis vectors

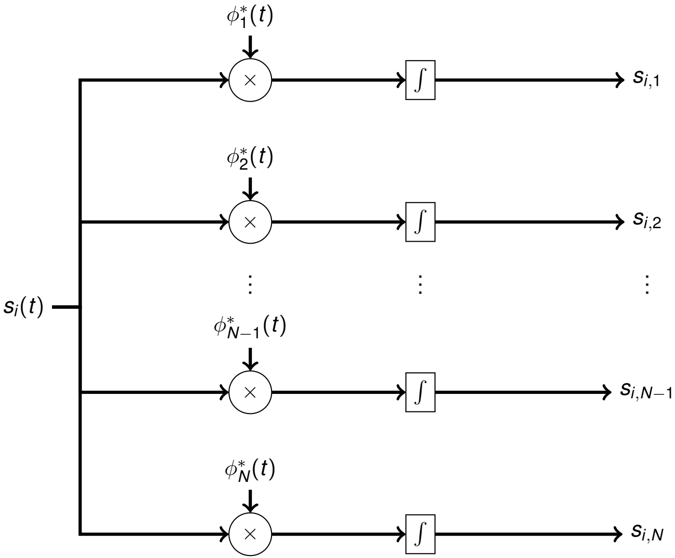
$$s_i(t) = \sum_{n=1}^{N} s_{i,n}\phi_n(t), \qquad i = 1, \ldots, M$$

- $s_i(t)$ is represented by the vector $\mathbf{s}_i = \begin{bmatrix} s_{i,1} & \cdots & s_{i,N} \end{bmatrix}^T$
- The set $\{\mathbf{s}_i : 1 \leq i \leq M\}$ is called the signal space representation or constellation

Constellation Point to Waveform



Waveform to Constellation Point



Gram-Schmidt Orthogonalization Procedure

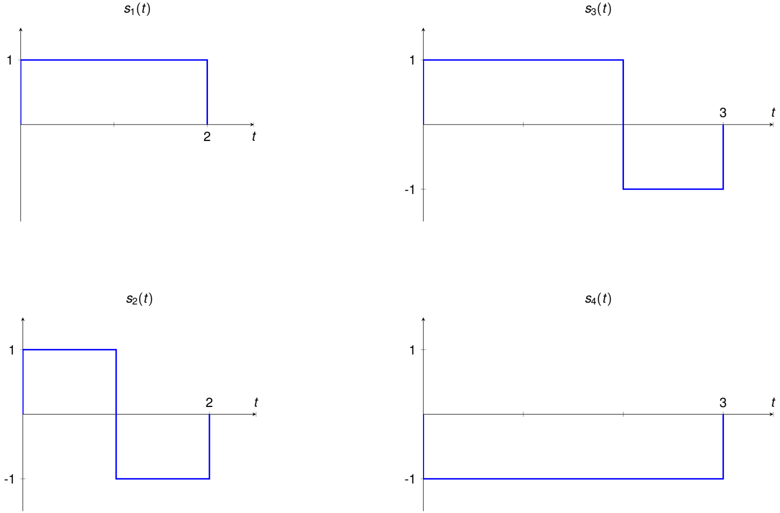
- Algorithm for calculating orthonormal basis
- Given $s_1(t), \ldots, s_M(t)$ the kth basis function is

$$\phi_k(t) = \frac{\gamma_k(t)}{\sqrt{E_k}}$$

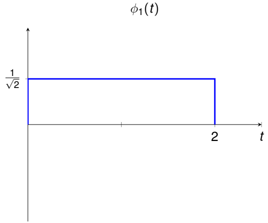
where

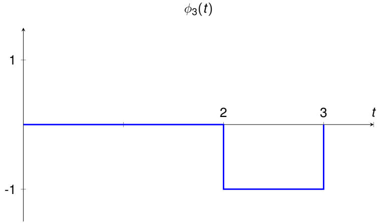
$$\begin{aligned} E_k &= \int_{-\infty}^{\infty} |\gamma_k(t)|^2 \, dt \\ \gamma_k(t) &= s_k(t) - \sum_{i=1}^{k-1} c_{k,i}\phi_i(t) \\ c_{k,i} &= \langle s_k(t), \phi_i(t) \rangle, \quad i = 1, 2, \ldots, k-1 \end{aligned}$$

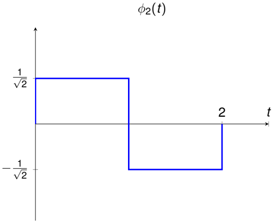
Gram-Schmidt Procedure Example



Gram-Schmidt Procedure Example







$$
\begin{aligned}
\mathbf{s}_1 &= \begin{bmatrix} \sqrt{2} & 0 & 0 \end{bmatrix}^T \\
\mathbf{s}_2 &= \begin{bmatrix} 0 & \sqrt{2} & 0 \end{bmatrix}^T \\
\mathbf{s}_3 &= \begin{bmatrix} \sqrt{2} & 0 & 1 \end{bmatrix}^T \\
\mathbf{s}_4 &= \begin{bmatrix} -\sqrt{2} & 0 & 1 \end{bmatrix}^T
\end{aligned}
$$

Properties of Signal Space Representation

- Energy

$$E_m = \int_{-\infty}^{\infty} |s_m(t)|^2 \, dt = \sum_{n=1}^{N} |s_{m,n}|^2 = \|\mathbf{s}_m\|^2$$

- Inner product

$$\langle s_i(t), s_j(t) \rangle = \langle \mathbf{s}_i, \mathbf{s}_j \rangle$$

Modulation Schemes

Pulse Amplitude Modulation

- Signal Waveforms

$$s_m(t) = A_m p(t), \quad 1 \leq m \leq M$$

where $p(t)$ is a pulse of duration T and A_m's denote the M possible amplitudes.

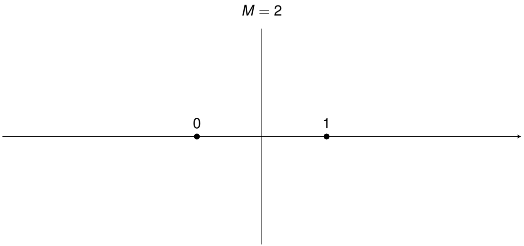
- Usually, $M = 2^k$ and amplitudes A_m take the values

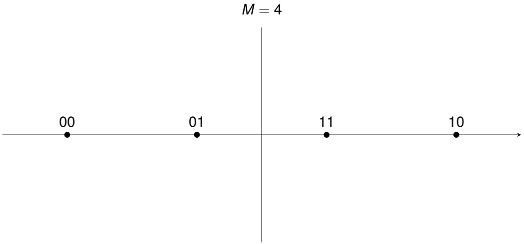
$$A_m = 2m - 1 - M, \quad 1 \leq m \leq M$$

Example (M=4) $A_1 = -3, A_2 = -1, A_3 = +1, A_4 = +3$

- Baseband PAM: $p(t)$ is a baseband signal
- Passband PAM: $p(t) = g(t) \cos 2\pi f_c t$ where $g(t)$ is baseband

Constellation for PAM





Phase Modulation

- Complex Envelope of Signals

$$s_m(t) = p(t)e^{j\frac{\pi(2m-1)}{M}}, \quad 1 \leq m \leq M$$

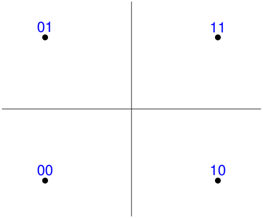
  where $p(t)$ is a real baseband pulse of duration T

- Passband Signals

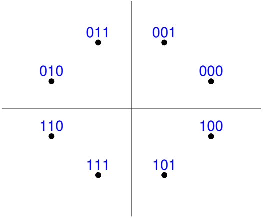
$$\begin{aligned} s_m^p(t) &= \text{Re}\left[\sqrt{2}s_m(t)e^{j2\pi f_c t}\right] \\ &= \sqrt{2}p(t)\cos\left(\frac{\pi(2m-1)}{M}\right)\cos 2\pi f_c t \\ &\quad -\sqrt{2}p(t)\sin\left(\frac{\pi(2m-1)}{M}\right)\sin 2\pi f_c t \end{aligned}$$

Constellation for PSK

QPSK, $M = 4$



Octal PSK, $M = 8$



Quadrature Amplitude Modulation

- Complex Envelope of Signals

$$s_m(t) = (A_{m,i} + jA_{m,q})p(t), \quad 1 \leq m \leq M$$

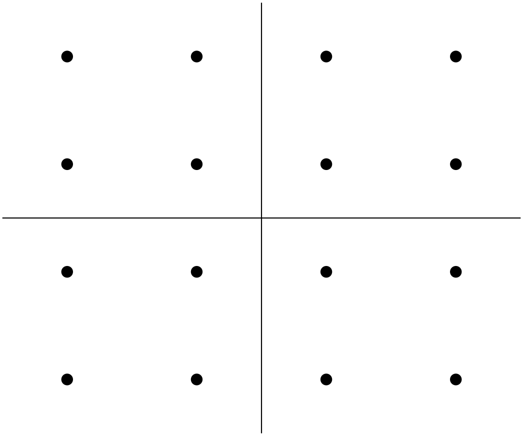
where $p(t)$ is a real baseband pulse of duration T

- Passband Signals

$$\begin{aligned} s_m^p(t) &= \mathrm{Re}\left[\sqrt{2}s_m(t)e^{j2\pi f_c t}\right] \\ &= \sqrt{2}A_{m,i}p(t)\cos 2\pi f_c t - \sqrt{2}A_{m,q}p(t)\sin 2\pi f_c t \end{aligned}$$

Constellation for QAM

16-QAM



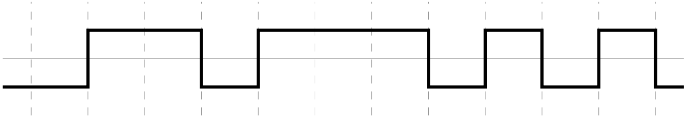
Power Spectral Density of Digitally Modulated Signals

PSD Definition for Digitally Modulated Signals

- Consider a real binary PAM signal

$$u(t) = \sum_{n=-\infty}^{\infty} b_n g(t - nT)$$

where $b_n = \pm 1$ with equal probability and $g(t)$ is a baseband pulse of duration T



- ~~PSD $= \mathcal{F}[R_u(\tau)]$~~ Not stationary or WSS

Cyclostationary Random Process

Definition (Cyclostationary RP)

A random process $X(t)$ is cyclostationary with respect to time interval T if it is statistically indistinguishable from $X(t - kT)$ for any integer k.

Definition (Wide Sense Cyclostationary RP)

A random process $X(t)$ is wide sense cyclostationary with respect to time interval T if the mean and autocorrelation functions satisfy

$$\begin{aligned} m_X(t) &= m_X(t - T) \quad \text{for all } t, \\ R_X(t_1, t_2) &= R_X(t_1 - T, t_2 - T) \quad \text{for all } t_1, t_2. \end{aligned}$$

Stationarizing a Cyclostationary Random Process

Theorem

Let $S(t)$ be a cyclostationary random process with respect to the time interval T. Suppose $D \sim U[0, T]$ and independent of $S(t)$. Then $S(t - D)$ is a stationary random process.

Proof Sketch

Let $V(t) = S(t - D)$. We prove that $V(t_1) \sim V(t_1 + \tau)$.

$$\begin{aligned}
P\left[V(t_1 + \tau) = v\right] &= \frac{1}{T}\int_0^T P\left[S(t_1 + \tau - x) = v\right]\, dx \\
&= \frac{1}{T}\int_{-\tau}^{T-\tau} P\left[S(t_1 - y) = v\right]\, dy \\
&= \frac{1}{T}\int_0^T P\left[S(t_1 - y) = v\right]\, dy \\
&= P\left[V(t_1) = v\right]
\end{aligned}$$

Stationarizing a Cyclostationary Random Process

Proof Sketch (Contd)

We prove that $V(t_1), V(t_2) \sim V(t_1+\tau), V(t_2+\tau)$.

$$
\begin{aligned}
& P\left[V(t_1+\tau) = v_1, V(t_2+\tau) = v_2\right] \\
&= \frac{1}{T}\int_0^T P\left[S(t_1+\tau-x) = v_1, S(t_2+\tau-x) = v_2\right]\, dx \\
&= \frac{1}{T}\int_{-\tau}^{T-\tau} P\left[S(t_1-y) = v_1, S(t_2-y) = v_2\right]\, dy \\
&= \frac{1}{T}\int_0^T P\left[S(t_1-y) = v_1, S(t_2-y) = v_2\right]\, dy \\
&= P\left[V(t_1) = v_1, V(t_2) = v_2\right]
\end{aligned}
$$

Stationarizing a Wide Sense Cyclostationary RP

Theorem

Let $S(t)$ be a wide sense cyclostationary RP with respect to the time interval T. Suppose $D \sim U[0, T]$ and independent of $S(t)$. Then $S(t - D)$ is a wide sense stationary RP.

Proof Sketch

Let $V(t) = S(t - D)$. We prove that $m_V(t)$ is a constant function.

$$m_V(t) = E\left[V(t)\right] = E\left[S(t-D)\right] = E\left[E\left[S(t-D)|D\right]\right]$$

$$E\left[S(t-D)|D = x\right] = E\left[S(t-x)\right] = m_S(t-x)$$

$$E\left[E\left[S(t-D)|D\right]\right] = \frac{1}{T}\int_0^T m_S(t-x)\,dx = \frac{1}{T}\int_0^T m_S(y)\,dy$$

Stationarizing a Wide Sense Cyclostationary RP

Proof Sketch (Contd)

We prove that $R_V(t_1, t_2)$ is a function of $t_1 - t_2 = kT + \epsilon$

$$
\begin{aligned}
R_V(t_1, t_2) &= E\left[V(t_1)V^*(t_2)\right] = E\left[S(t_1 - D)S^*(t_2 - D)\right] \\
&= \frac{1}{T}\int_0^T R_S(t_1 - x, t_2 - x)\, dx \\
&= \frac{1}{T}\int_0^T R_S(t_1 - kT - x, t_2 - kT - x)\, dx \\
&= \frac{1}{T}\int_{-\epsilon}^{T-\epsilon} R_S(t_1 - kT - \epsilon - y, t_2 - kT - \epsilon - y)\, dy \\
&= \frac{1}{T}\int_{-\epsilon}^{T-\epsilon} R_S(t_1 - t_2 - y, -y)\, dy \\
&= \frac{1}{T}\int_0^T R_S(t_1 - t_2 - y, -y)\, dy
\end{aligned}
$$

Power Spectral Density of a Realization

Time windowed realizations have finite energy

$$
\begin{aligned}
x_{T_o}(t) &= x(t) I_{[-\frac{T_o}{2}, \frac{T_o}{2}]}(t) \\
S_{T_o}(f) &= \mathcal{F}(x_{T_o}(t)) \\
\hat{S}_x(f) &= \frac{|S_{T_o}(f)|^2}{T_o} \quad \text{(PSD Estimate)}
\end{aligned}
$$

PSD of a realization

$$
\bar{S}_x(f) = \lim_{T_o \to \infty} \frac{|S_{T_o}(f)|^2}{T_o}
$$

$$
\frac{|S_{T_o}(f)|^2}{T_o} \rightleftharpoons \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u) x_{T_o}^*(u - \tau)\, du = \hat{R}_s(\tau)
$$

Power Spectral Density of a Cyclostationary Process

$S(t)S^*(t-\tau) \sim S(t+T)S^*(t+T-\tau)$ for cyclostationary $S(t)$

$$\begin{aligned}
\hat{R}_s(\tau) &= \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} s(t)s^*(t-\tau)\,dt \\
&= \frac{1}{KT}\int_{-\frac{KT}{2}}^{\frac{KT}{2}} s(t)s^*(t-\tau)\,dt \qquad \text{for } T_o = KT \\
&= \frac{1}{T}\int_0^T \frac{1}{K}\sum_{k=-\frac{K}{2}}^{\frac{K}{2}} s(t+kT)s^*(t+kT-\tau)\,dt \\
&\to \frac{1}{T}\int_0^T E\left[S(t)S^*(t-\tau)\right]\,dt \\
&= \frac{1}{T}\int_0^T R_S(t,t-\tau)\,dt = R_V(\tau)
\end{aligned}$$

PSD of a cyclostationary process = $\mathcal{F}[R_V(\tau)]$

Power Spectral Density of a Cyclostationary Process

To obtain the PSD of a cyclostationary process

- Stationarize it
- Calculate autocorrelation function of stationarized process
- Calculate Fourier transform of autocorrelation

or

- Calculate autocorrelation of cyclostationary process $R_S(t, t-\tau)$
- Average autocorrelation between 0 and T, $R_S(\tau) = \frac{1}{T}\int_0^T R_S(t, t-\tau)\, dt$
- Calculate Fourier transform of averaged autocorrelation $R_S(\tau)$

Power Spectral Density of Linearly Modulated Signals

PSD of a Linearly Modulated Signal

- Consider

$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

- $u(t)$ is cyclostationary wrt to T if $\{b_n\}$ is stationary
- $u(t)$ is wide sense cyclostationary wrt to T if $\{b_n\}$ is WSS
- Suppose $R_b[k] = E[b_n b_{n-k}^*]$
- Let $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$
- The PSD of $u(t)$ is given by

$$S_u(f) = S_b\left(e^{j2\pi fT}\right) \frac{|P(f)|^2}{T}$$

PSD of a Linearly Modulated Signal

$$
\begin{aligned}
R_u(\tau) \\
&= \frac{1}{T}\int_0^T R_u(t+\tau,t)\,dt \\
&= \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[b_n b_m^* p(t-nT+\tau)p^*(t-mT)\right]\,dt \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-mT}^{-(m-1)T} E\left[b_{m+k} b_m^* p(u-kT+\tau)p^*(u)\right]\,du \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[b_{m+k} b_m^* p(u-kT+\tau)p^*(u)\right]\,du \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} R_b[k]\int_{-\infty}^{\infty} p(u-kT+\tau)p^*(u)\,du
\end{aligned}
$$

PSD of a Linearly Modulated Signal

$$R_u(\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} R_b[k] \int_{-\infty}^{\infty} p(u - kT + \tau) p^*(u)\, du$$

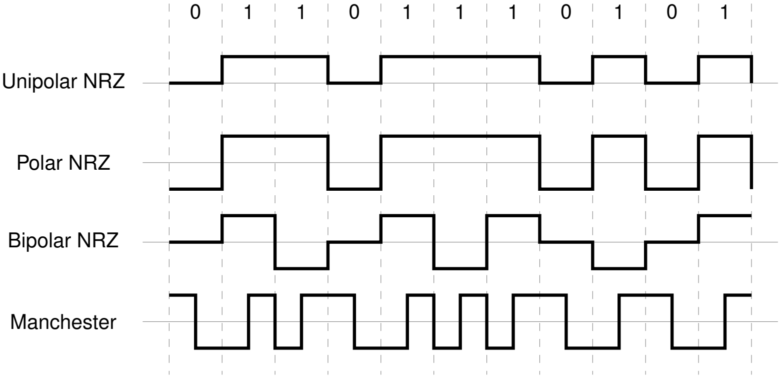
$$\begin{aligned} \int_{-\infty}^{\infty} p(u + \tau) p^*(u)\, du \quad &\rightleftharpoons \quad |P(f)|^2 \\ \int_{-\infty}^{\infty} p(u - kT + \tau) p^*(u)\, du \quad &\rightleftharpoons \quad |P(f)|^2 e^{-j2\pi fkT} \end{aligned}$$

$$\begin{aligned} S_u(f) = \mathcal{F}\left[R_u(\tau)\right] \quad &= \quad \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi fkT} \\ &= \quad S_b\left(e^{j2\pi fT}\right) \frac{|P(f)|^2}{T} \end{aligned}$$

where $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$.

Power Spectral Density of Line Codes

Line Codes



Further reading: *Digital Communications*, Simon Haykin, Chapter 6

Unipolar NRZ

- Symbols independent and equally likely to be 0 or A

$$P(b[n] = 0) = P(b[n] = A) = \frac{1}{2}$$

- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} \frac{A^2}{2} & k = 0 \\ \frac{A^2}{4} & k \neq 0 \end{cases}$$

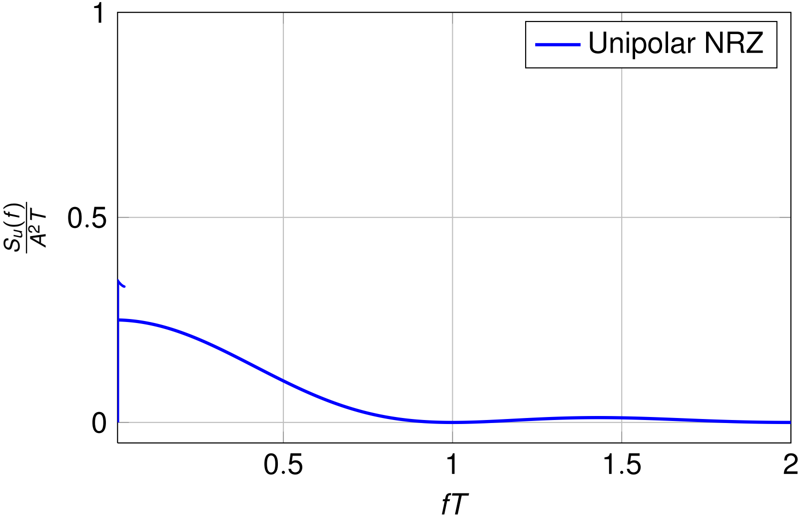
- $p(t) = I_{[0,T)}(t) \Rightarrow P(f) = T\text{sinc}(fT)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f) = \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi kfT}$$

Unipolar NRZ

$$
\begin{aligned}
S_u(f) &= \frac{A^2T}{4}\text{sinc}^2(fT) + \frac{A^2T}{4}\text{sinc}^2(fT)\sum_{k=-\infty}^{\infty} e^{-j2\pi kfT} \\
&= \frac{A^2T}{4}\text{sinc}^2(fT) + \frac{A^2}{4}\text{sinc}^2(fT)\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \\
&= \frac{A^2T}{4}\text{sinc}^2(fT) + \frac{A^2}{4}\delta(f)
\end{aligned}
$$

Normalized PSD plot



Polar NRZ

- Symbols independent and equally likely to be $-A$ or A

$$P\left(b[n] = -A\right) = P\left(b[n] = A\right) = \frac{1}{2}$$

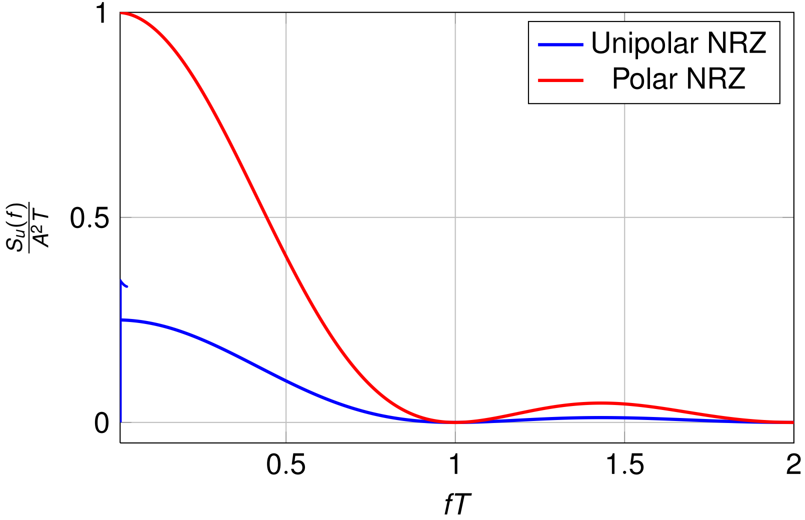
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

- $P(f) = T\text{sinc}(fT)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f) = A^2 T\text{sinc}^2(fT)$$

Normalized PSD plots



Manchester

- Symbols independent and equally likely to be $-A$ or A

$$P\left(b[n] = -A\right) = P\left(b[n] = A\right) = \frac{1}{2}$$

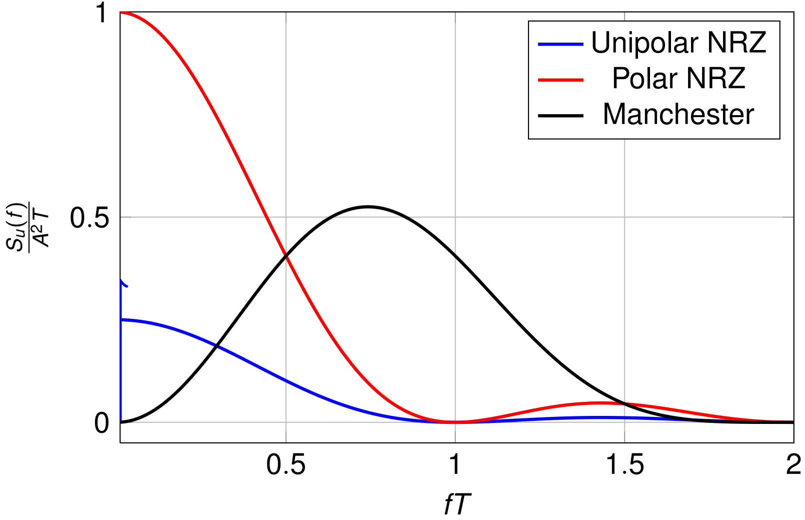
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

- $P(f) = jT\text{sinc}\left(\frac{fT}{2}\right)\sin\left(\frac{\pi fT}{2}\right)$
- Power Spectral Density

$$S_u(f) = A^2 T\text{sinc}^2\left(\frac{fT}{2}\right)\sin^2\left(\frac{\pi fT}{2}\right)$$

Normalized PSD plots



Bipolar NRZ

- Successive 1's have alternating polarity

$$
\begin{aligned}
0 &\rightarrow \text{Zero amplitude} \\
1 &\rightarrow +A \text{ or } -A
\end{aligned}
$$

- Probability mass function of $b[n]$

$$
\begin{aligned}
P\left(b[n] = 0\right) &= \frac{1}{2} \\
P\left(b[n] = -A\right) &= \frac{1}{4} \\
P\left(b[n] = A\right) &= \frac{1}{4}
\end{aligned}
$$

- Symbols are identically distributed but they are not independent

Bipolar NRZ

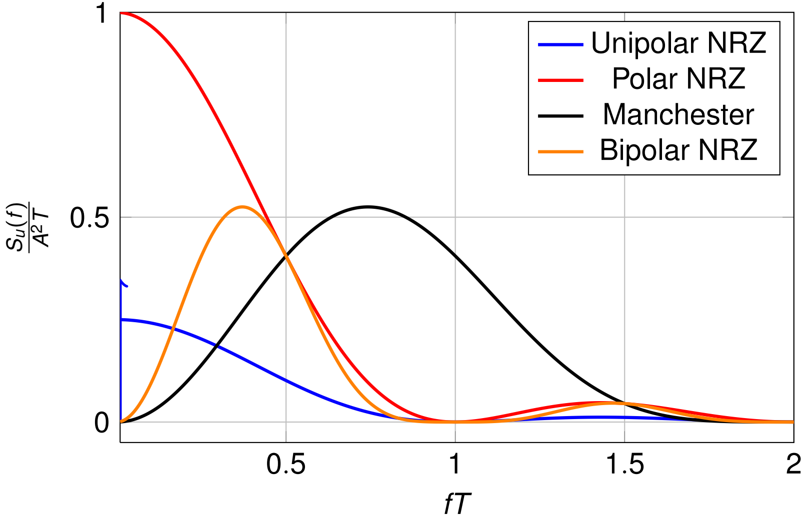
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2/2 & k = 0 \\ -A^2/4 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

- Power Spectral Density

$$\begin{aligned} S_u(f) &= T\text{sinc}^2(fT)\left[\frac{A^2}{2} - \frac{A^2}{4}\left(e^{j2\pi fT} + e^{-j2\pi fT}\right)\right] \\ &= \frac{A^2 T}{2}\text{sinc}^2(fT)\left[1 - \cos(2\pi fT)\right] \\ &= A^2 T\text{sinc}^2(fT)\sin^2(\pi fT) \end{aligned}$$

Normalized PSD plots



Thanks for your attention