Carrier Phase and Symbol Timing Synchronization

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The System Model

- Consider the following complex baseband signal $s(t)$

$$s(t) = \sum_{i=0}^{K-1} b_i p(t - iT)$$

where b_i's are complex symbols

- Suppose the LO frequency at the transmitter is f_c

$$s_p(t) = \text{Re}\left[\sqrt{2}s(t)e^{j2\pi f_c t}\right].$$

- Suppose that the LO frequency at the receiver is $f_c - \Delta f$
- The received passband signal is

$$y_p(t) = As_p(t - \tau) + n_p(t)$$

- The complex baseband representation of the received signal is then

$$y(t) = Ae^{j(2\pi \Delta f t + \theta)} s(t - \tau) + n(t)$$

The System Model

$$y(t) = Ae^{j(2\pi\Delta ft+\theta)} \sum_{i=0}^{K-1} b_i p(t - iT - \tau) + n(t)$$

- Assume that the receiver side symbol rate is $\frac{1+\delta}{T}$
- The unknown parameters are A, τ, θ, Δf and δ

  Timing Synchronization Estimation of τ

  Carrier Synchronization Estimation of θ and Δf

  Clock Synchronization Estimation of δ
- Estimation approach depends on knowledge of b_i's
  - Data-Aided Approach The b_i's are known
    - The preamble of a packet contains known symbols
  - Decision-Directed Approach Decisions of b_i's are used
    - Effective when symbol error rate is low
  - Non-Decision-Directed Approach The b_i's are unknown
    - Averaging over the symbol distribution

Likelihood Function of Signals in AWGN

- The likelihood function of signals in real AWGN is

$$L(y|s_\phi) = \exp\left(\frac{1}{\sigma^2}\left[\langle y, s_\phi\rangle - \frac{\|s_\phi\|^2}{2}\right]\right)$$

- The likelihood function of signals in complex AWGN is

$$L(y|s_\phi) = \exp\left(\frac{1}{\sigma^2}\left[\mathrm{Re}(\langle y, s_\phi\rangle) - \frac{\|s_\phi\|^2}{2}\right]\right)$$

- Maximizing these likelihood functions as functions of ϕ results in the ML estimator

Carrier Phase Estimation

- The change in phase due to the carrier offset Δf is $2\pi\Delta f T$ in a symbol interval T
- The phase can be assumed to be constant over multiple symbol intervals
- Assume that the phase θ is the only unknown parameter
- Assume that $s(t)$ is a known signal in the following

$$y(t) = s(t)e^{j\theta} + n(t)$$

- The likelihood function for this scenario is given by

$$L(y|s_\theta) = \exp\left(\frac{1}{\sigma^2}\left[\mathrm{Re}(\langle y, se^{j\theta}\rangle) - \frac{\|se^{j\theta}\|^2}{2}\right]\right)$$

- Let $\langle y, s\rangle = Z = |Z|e^{j\phi} = Z_c + jZ_s$

$$\begin{aligned} \langle y, se^{j\theta}\rangle &= e^{-j\theta}Z = |Z|e^{j(\phi-\theta)} \\ \mathrm{Re}(\langle y, se^{j\theta}\rangle) &= |Z|\cos(\phi-\theta) \\ \|se^{j\theta}\|^2 &= \|s\|^2 \end{aligned}$$

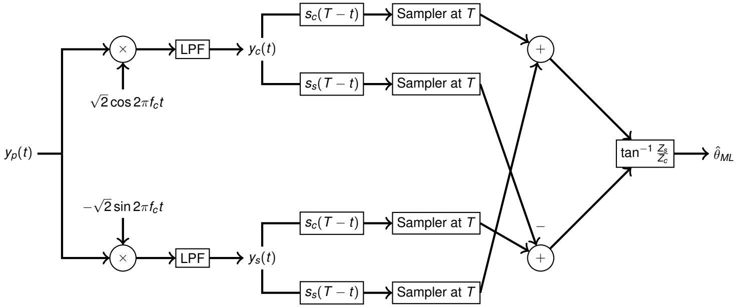
Carrier Phase Estimation

- The likelihood function for this scenario is given by

$$L(y|s_\theta) = \exp\left(\frac{1}{\sigma^2}\left[|Z|\cos(\phi-\theta) - \frac{\|s\|^2}{2}\right]\right)$$

- The ML estimate of θ is given by

$$\hat{\theta}_{ML} = \phi = \arg(\langle y, s\rangle) = \tan^{-1}\frac{Z_s}{Z_c}$$



Phase Locked Loop

- The carrier offset will cause the phase to change slowly
- A tracking mechanism is required to track the changes in phase
- For simplicity, consider an unmodulated carrier

$$y_p(t) = A\cos(2\pi f_c t + \theta) + n(t)$$

- The log likelihood function for this scenario is given by

$$\begin{aligned} &\ln L(y|s_\theta) \\ &= \frac{1}{\sigma^2}\left[\langle y_p(t), A\cos(2\pi f_c t + \theta)\rangle - \frac{\|A\cos(2\pi f_c t + \theta)\|^2}{2}\right] \end{aligned}$$

- For an observation interval T_o, we get $\hat{\theta}_{ML}$ by maximizing

$$\Lambda(\theta) = \frac{A}{\sigma^2}\int_{T_o} y_p(t)\cos(2\pi f_c t + \theta)\,dt$$

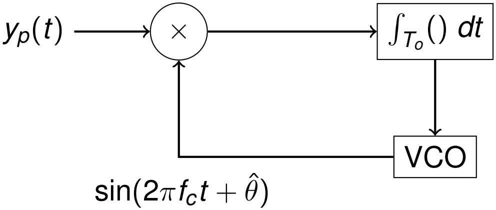
Phase Locked Loop

- A necessary condition for a maximum at $\hat{\theta}_{ML}$ is

$$\frac{\partial}{\partial \theta} \Lambda(\hat{\theta}_{ML}) = 0$$

- This implies

$$\int_{T_o} y_p(t) \sin(2\pi f_c t + \hat{\theta}_{ML})\, dt = 0$$



Non-Decision-Directed PLL for BPSK

- When the symbols are unknown we average the likelihood function over the symbol distribution
- Suppose the transmitted signal is given by

$$s(t) = A\cos(2\pi f_c t + \theta), \quad 0 \le t \le T$$

where A is equally likely to be ± 1. The likelihood function is given by

$$L(r|\theta) = \exp\left(\frac{1}{\sigma^2}\left[\int_0^T r(t)s(t)dt - \frac{\|s(t)\|^2}{2}\right]\right)$$

- Neglecting the energy of the signal as it is parameter independent we get the likelihood function

$$\Lambda(\theta) = \exp\left(\frac{1}{\sigma^2}\int_0^T r(t)s(t)dt\right)$$

Non-Decision-Directed PLL for BPSK

- We have to average $\Lambda(\theta)$ over the distribution of A

$$
\begin{aligned}
\bar{\Lambda}(\theta) &= E_A\left[\Lambda(\theta)\right] \\
&= \frac{1}{2}\exp\left[\frac{1}{\sigma^2}\int_0^T r(t)\cos(2\pi f_c t+\theta)\,dt\right] \\
&\quad +\frac{1}{2}\exp\left[-\frac{1}{\sigma^2}\int_0^T r(t)\cos(2\pi f_c t+\theta)\,dt\right] \\
&= \cosh\left[\frac{1}{\sigma^2}\int_0^T r(t)\cos(2\pi f_c t+\theta)\,dt\right]
\end{aligned}
$$

Non-Decision-Directed PLL for BPSK

- To find $\hat{\theta}_{ML}$ we can maximize $\ln \bar{\Lambda}(\theta)$ instead of $\bar{\Lambda}(\theta)$

$$\ln \bar{\Lambda}(\theta) = \ln \cosh \left[\frac{1}{\sigma^2} \int_0^T r(t) \cos(2\pi f_c t + \theta)\, dt \right]$$

- Maximizing this function is difficult but approximations can be made which make the maximization easy

$$\ln \cosh x = \begin{cases} \frac{x^2}{2}, & |x| \ll 1 \\ |x|, & |x| \gg 1 \end{cases}$$

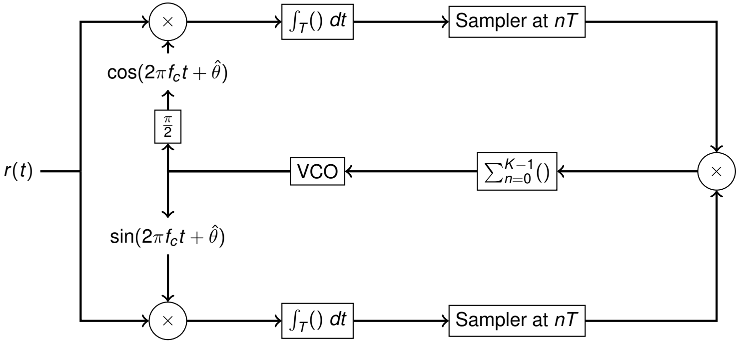
- For an observation over K independent symbols

$$\bar{\Lambda}_K(\theta) = \exp \left\{ \sum_{n=0}^{K-1} \left[\frac{1}{\sigma^2} \int_{nT}^{(n+1)T} r(t) \cos(2\pi f_c t + \theta)\, dt \right]^2 \right\}$$

Non-Decision-Directed PLL for BPSK

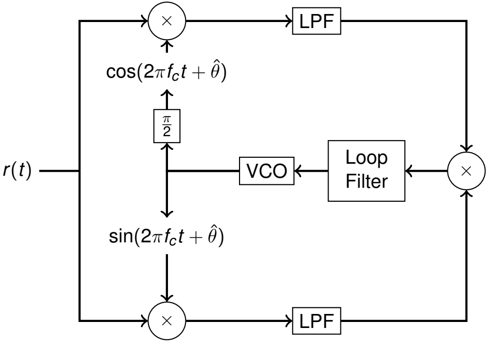
A necessary condition on the ML estimate $\hat{\theta}_{ML}$ is

$$\sum_{n=0}^{K-1} \int_{nT}^{(n+1)T} r(t)\cos(2\pi f_c t + \hat{\theta}_{ML})\, dt \times \int_{nT}^{(n+1)T} r(t)\sin(2\pi f_c t + \hat{\theta}_{ML})\, dt = 0$$



Costas Loop

- Developed by Costas in 1956



- The received signal is

$$\begin{aligned} r(t) &= A(t)\cos(2\pi f_c t + \theta) + n(t) \\ &= s(t) + n(t) \end{aligned}$$

Costas Loop

- The input to the loop filter is $e(t) = y_c(t)y_s(t)$ where

$$
\begin{aligned}
y_c(t) &= \text{LPF}\left\{[s(t)+n(t)]\cos(2\pi f_c t + \hat{\theta})\right\} \\
&= \frac{1}{2}[A(t)+n_i(t)]\cos\Delta\theta + \frac{1}{2}n_q(t)\sin\Delta\theta \\
y_s(t) &= \text{LPF}\left\{[s(t)+n(t)]\sin(2\pi f_c t + \hat{\theta})\right\} \\
&= \frac{1}{2}[A(t)+n_i(t)]\sin\Delta\theta - \frac{1}{2}n_q(t)\cos\Delta\theta
\end{aligned}
$$

where

$$
\begin{aligned}
n_i(t) &= \text{LPF}\left\{n(t)\cos(2\pi f_c t + \theta)\right\} \\
n_q(t) &= \text{LPF}\left\{n(t)\sin(2\pi f_c t + \theta)\right\}
\end{aligned}
$$

Costas Loop

- The input to the loop filter is given by

$$\begin{aligned} e(t) &= \frac{1}{8}\left\{[A(t)+n_i(t)]^2 - n_q^2(t)\right\}\sin(2\Delta\theta) \\ &\quad -\frac{1}{4}n_q(t)\,[A(t)+n_i(t)]\cos(2\Delta\theta) \\ &= \frac{1}{8}A^2(t)\sin(2\Delta\theta) + \text{noise}\times\text{signal} + \text{noise}\times\text{noise} \end{aligned}$$

- The VCO output has a 180° ambiguity necessitating differential encoding of data

Symbol Timing Estimation

- Consider the complex baseband received signal

$$y(t) = As(t-\tau)e^{j\theta} + n(t)$$

  where A, τ and θ are unknown and $s(t)$ is known
- For $\Gamma = [\tau, \theta, A]$ the likelihood function is

$$L(y|s_\Gamma) = \exp\left(\frac{1}{\sigma^2}\left[\mathrm{Re}\left(\langle y, s_\Gamma\rangle\right) - \frac{\|s_\Gamma\|^2}{2}\right]\right)$$

- For a large enough observation interval, the signal energy does not depend on τ and $\|s_\Gamma\|^2 = A^2\|s\|^2$
- For $s_{MF}(t) = s^*(-t)$ we have

$$\begin{aligned}\langle y, s_\Gamma\rangle &= Ae^{-j\theta}\int y(t)s^*(t-\tau)\,dt \\ &= Ae^{-j\theta}\int y(t)s_{MF}(\tau-t)\,dt \\ &= Ae^{-j\theta}(y \star s_{MF})(\tau)\end{aligned}$$

Symbol Timing Estimation

- Maximizing the likelihood function is equivalent to maximizing the following cost function

$$J(\tau, A, \theta) = \text{Re}\left(Ae^{-j\theta}(y \star s_{MF})(\tau)\right) - \frac{A^2\|s\|^2}{2}$$

- For $(y \star s_{MF})(\tau) = Z(\tau) = |Z(\tau)|e^{j\phi(\tau)}$ we have

$$\text{Re}\left(Ae^{-j\theta}(y \star s_{MF})(\tau)\right) = A|Z(\tau)|\cos(\phi(\tau) - \theta)$$

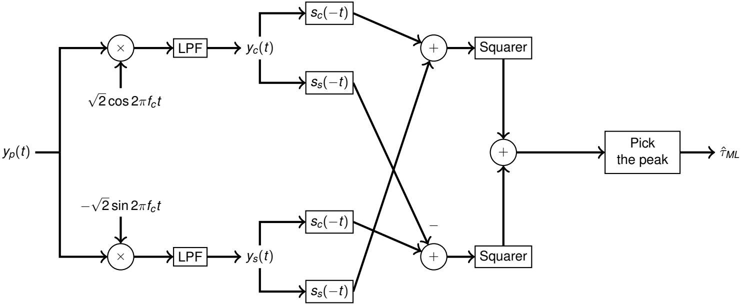
- The maximizing value of θ is equal to $\phi(\tau)$
- Substituting this value of θ gives us the following cost function

$$J(\tau, A) = \underset{\theta}{\text{argmax}}\, J(\tau, A, \theta) = A|(y \star s_{MF})(\tau)| - \frac{A^2\|s\|^2}{2}$$

Symbol Timing Estimation

- The ML estimator of the delay picks the peak of the matched filter output

$$\hat{\tau}_{ML} = \underset{\tau}{\text{argmax}} \left| (y \star s_{MF})(\tau) \right|$$



Decision-Directed Symbol Timing Tracking

- For illustration, consider a baseband PAM signal[1]

$$r(t) = \sum_i b_i p(t - iT - \tau) + n(t)$$

  where τ is unknown and $p(t)$ is known
- Suppose the decisions on the b_i's are correct
- For $s_\tau(t) = \sum_i b_i p(t - iT - \tau)$ the likelihood function is

$$L(r|s_\tau) = \exp\left(\frac{1}{\sigma^2}\left[\langle r, s_\tau \rangle - \frac{\|s_\tau\|^2}{2}\right]\right)$$

- For a large enough observation interval T_o, the signal energy can be assumed to be independent of τ

[1] Complex baseband case is only slightly different

Decision-Directed Symbol Timing Tracking

- The ML estimate of τ is obtained by maximizing

$$\begin{aligned}\Lambda(\tau) &= \int_{T_o} r(t)s_\tau(t)\,dt \\ &= \sum_i b_i \int_{T_o} r(t)p(t-iT-\tau)\,dt = \sum_i b_i y(iT+\tau)\end{aligned}$$

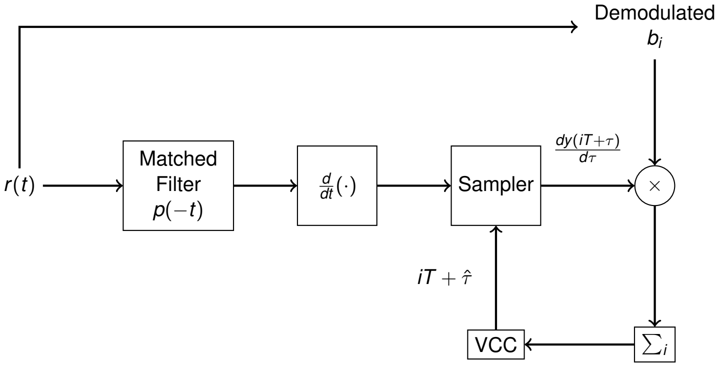
  where

$$y(\alpha) = \int_{T_o} r(t)p(t-\alpha)\,dt$$

- A necessary condition on $\hat{\tau}_{ML}$ is

$$\frac{d}{d\tau}\Lambda(\hat{\tau}_{ML}) = \sum_i b_i \frac{dy(iT+\hat{\tau}_{ML})}{d\tau} = 0$$

Decision-Directed Symbol Timing Tracking



$$\sum_i b_i \frac{dy(iT + \hat{\tau}_{ML})}{d\tau} = 0$$

Non-Decision-Directed Symbol Timing Tracking

- When the symbols are unknown we average the likelihood function over the symbol distribution
- Suppose the transmitted signal is binary PAM

$$r(t) = \sum_i b_i p(t - iT - \tau) + n(t)$$

where the b_i's are equally likely to be ± 1.

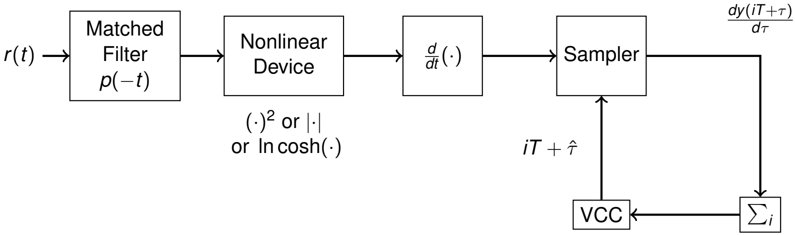
- The ML estimate of τ is obtained by maximizing the average of the log-likelihood function

$$\bar{\Lambda}(\tau) = \sum_i \ln \cosh[y(iT + \tau)]$$

where

$$y(\alpha) = \int_{T_o} r(t) p(t - \alpha)\, dt$$

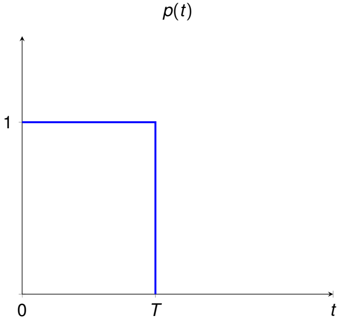
Non-Decision-Directed Symbol Timing Tracking

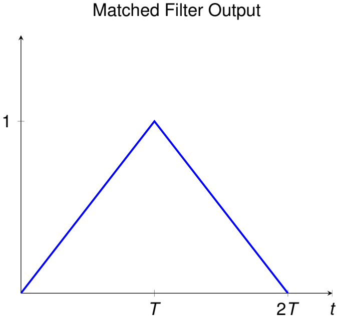


$$\sum_i \frac{d}{d\tau} \ln\cosh[y(iT + \hat{\tau}_{ML})] = 0$$

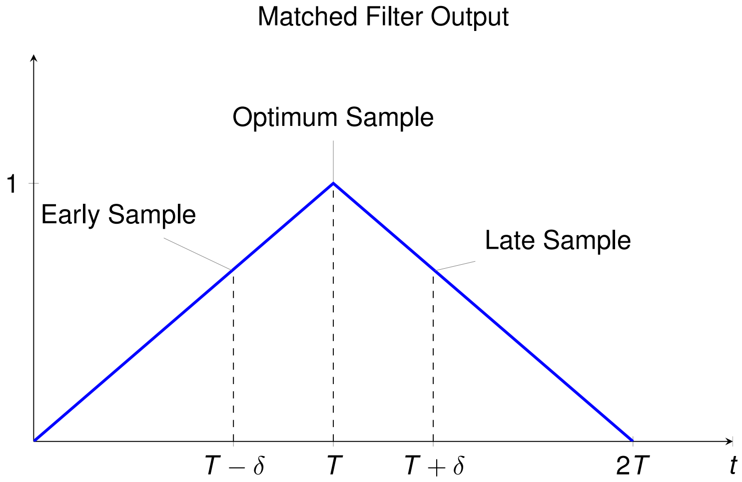
Early-Late Gate Synchronizer

- Non-decision directed timing tracker which exploits symmetry in matched filter output



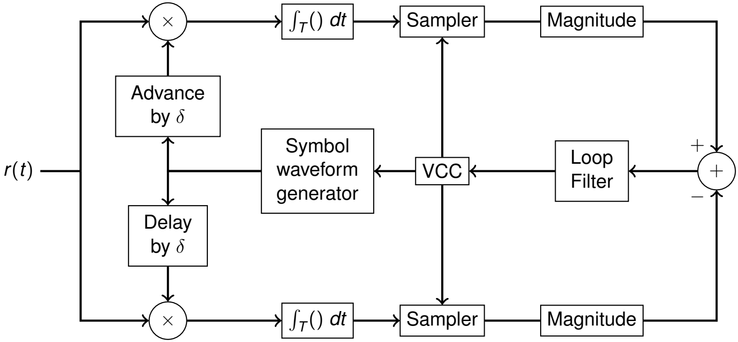


Early-Late Gate Synchronizer



- The values of the early and late samples are equal

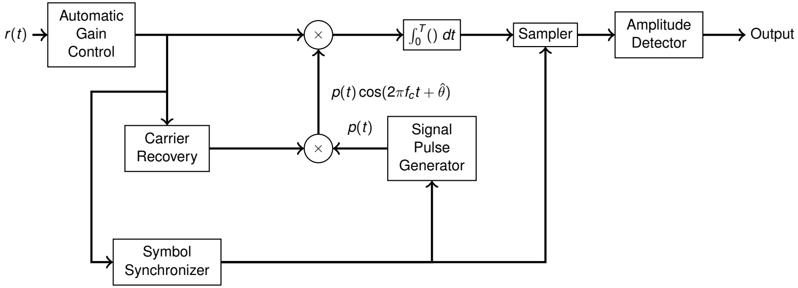
Early-Late Gate Synchronizer



- The motivation for this structure can be seen from the following approximation

$$\frac{d\Lambda(\tau)}{d\tau} \approx \frac{\Lambda(\tau+\delta) - \Lambda(\tau-\delta)}{2\delta}$$

Block Diagram of *M*-ary PAM Receiver



Thanks for your attention