EE 703: Digital Message Transmission

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Endsem Exam: **45 points** Date: November 18, 2013

Each question is worth 5 points.

1. Consider the following binary hypothesis testing problem where the hypotheses are equally likely.

$$\begin{aligned} H_0 &: \; Y \sim U\left[-\sqrt{\frac{e^2\pi}{2}}, \sqrt{\frac{e^2\pi}{2}}\right] \\ H_1 &: \; Y \sim \mathcal{N}(0,1) \end{aligned}$$

   U denotes the uniform distribution, $\mathcal{N}$ denotes the Gaussian distribution and e is the base of the natural logarithm.

   (a) Find the decision error probability of the rule which decides H_1 is true if $|Y| > \sqrt{\frac{e^2\pi}{2}}$ and decides H_0 is true if $|Y| \leq \sqrt{\frac{e^2\pi}{2}}$. Express your answer in terms of the Q function.

   (b) Find the decision error probability of the optimal decision rule. Express your answer in terms of the Q function.

2. Suppose a binary source is equally likely to emit 0 or 1. Suppose the bit 0 is mapped to 101, the bit 1 is mapped to 010 and the three bits are sent over a binary symmetric channel (BSC) with crossover probability $p < \frac{1}{2}$. The output of the BSC is used to decide which bit was emitted by the source.

   (a) Find the optimal decision rule.

   (b) Find the decision error probability of the optimal decision rule in terms of p.

3. Prove that $\mathbf{U} = e^{j\phi}\mathbf{Z}$ is a complex Gaussian vector when $\mathbf{Z}$ is a complex Gaussian vector.

4. For $M = 2^b$, suppose M **orthogonal** real signals $s_i(t),\ i = 1, \ldots, M$ are used for transmitting b bits over a real AWGN channel with PSD $\frac{N_0}{2}$. If all the signals have the same energy E and are equally likely to be transmitted, derive the following as a function of E, N_0, b or M when the optimal receiver is used.

   (a) The power efficiency of this modulation scheme

   (b) The union bound on the symbol error probability

   (c) The nearest neighbor approximation of the symbol error probability

5. Consider the following signals.

$$s_1(t) = -3Ap(t), s_2(t) = -Ap(t), s_3(t) = Ap(t), s_4(t) = 3Ap(t)$$

   where $p(t) = I_{[0,1]}(t)$. If these signals are equally likely to be sent over a real AWGN channel with power spectral density $\frac{N_0}{2}$, derive the following when the optimal receiver is used.

   (a) The power efficiency of this modulation scheme.

   (b) The exact symbol error probability as a function of E_b and N_0

   (c) The union bound on the symbol error probability

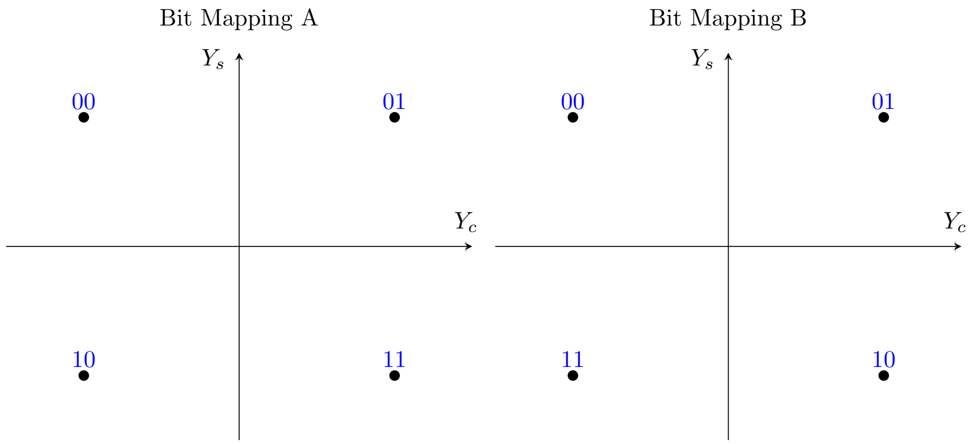
   (d) The intelligent union bound on the symbol error probability

   (e) The nearest neighbor approximation of the symbol error probability

6. Derive the bit error rate of the ML receiver for QPSK for the two symbol to bit mappings shown below. Let s_i be the symbol in the ith quadrant given by

$$s_1 = \sqrt{E_b} + j\sqrt{E_b}, s_2 = -\sqrt{E_b} + j\sqrt{E_b}, s_3 = -\sqrt{E_b} - j\sqrt{E_b}, s_4 = \sqrt{E_b} - j\sqrt{E_b}.$$

Assume the transmitted symbols are corrupted by circularly symmetric complex Gaussian noise with variance $2\sigma^2 = N_0$.



7. Suppose observations X_i and Y_i ($i = 1, \ldots, N$) depend on an unknown parameter A as per the following distributions.

$$X_i \sim \mathcal{N}(A, \sigma^2), \quad i = 1, 2, \ldots, N$$
$$Y_i \sim \mathcal{N}(A, 2\sigma^2), \quad i = 1, 2, \ldots, N$$

Note that the variance of Y_i is twice the variance of X_i. Assume that X_i and X_j are independent for $i \neq j$. Assume that Y_i and Y_j are independent for $i \neq j$. Assume that X_i and Y_j are independent for all i, j. Assume σ^2 is known.

(a) Derive the ML estimator for A.

(b) Find the mean and variance of the ML estimate.

8. Consider the real baseband received signal

$$y(t) = As(t - \tau) + n(t)$$

where $n(t)$ is a real WGN process with known PSD σ^2. The parameters A and τ are unknown while $s(t)$ is known. Derive the ML estimates of A and τ. Recall that the likelihood ratio of signals in real AWGN is

$$L(y|s_\theta) = \exp\left(\frac{1}{\sigma^2}\left[\langle y, s_\theta\rangle - \frac{\|s_\theta\|^2}{2}\right]\right)$$

where s_θ is the signal containing the unknown parameters θ.

9. Suppose you observe a real white Gaussian noise process with unknown power spectral density σ^2. Devise a scheme to estimate σ^2. Your estimator need not be the ML estimator but it should satisfy the following desirable properties.

(a) The mean of the estimator should be equal to σ^2.

(b) You should be able to make the variance of your estimator smaller by varying some parameters of your scheme.

You need to provide proofs that the above properties are satisfied for the estimator you propose.

Hint: The fourth moment of a Gaussian random variable with mean μ and variance ν^2 is $\mu^4 + 6\mu^2\nu^2 + 3\nu^4$.