The Fourier Transform

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Definition

- Fourier transform

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt$$

- Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft}\, df$$

- Notation

$$x(t) \rightleftharpoons X(f)$$

Properties of Fourier Transform

- Linearity

$$ax_1(t) + bx_2(t) \rightleftharpoons aX_1(f) + bX_2(f)$$

- Duality

$$X(t) \rightleftharpoons x(-f)$$

- Conjugacy

$$x^*(t) \rightleftharpoons X^*(-f)$$

- Time scaling

$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Properties of Fourier Transform

- Time shift

$$x(t - t_0) \rightleftharpoons e^{-j2\pi f t_0} X(f)$$

- Modulation

$$x(t) e^{j2\pi f_0 t} \rightleftharpoons X(f - f_0)$$

- Convolution

$$x(t) \star y(t) \rightleftharpoons X(f) Y(f)$$

- Multiplication

$$x(t) y(t) \rightleftharpoons X(f) \star Y(f)$$

Dirac Delta Function

- Zero for nonzero arguments

$$\delta(t) = 0, \quad \forall t \neq 0$$

- Unit area

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

- Sifting property

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_0)\, dt = x(t_0)$$

- Fourier transform

$$\delta(t) \rightleftharpoons \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft}\, dt = 1$$

Fourier Transforms using Dirac Function

- DC Signal

$$1 \rightleftharpoons \delta(f)$$

- Complex Exponential

$$e^{j2\pi f_c t} \rightleftharpoons \delta(f - f_c)$$

- Sinusoidal Functions

$$\begin{aligned} \cos(2\pi f_c t) &\rightleftharpoons \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)] \\ \sin(2\pi f_c t) &\rightleftharpoons \frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)] \end{aligned}$$

Properties of Fourier Transform

- Parseval's theorem

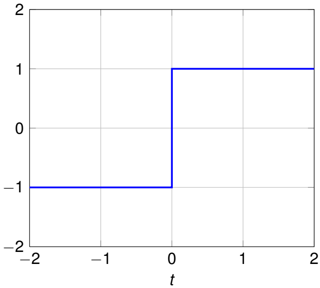
$$\int_{-\infty}^{\infty} x(t)y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\, df$$

- Rayleigh's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$$

Signum Function

$$\mathrm{sgn}(t) = \begin{cases} +1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$

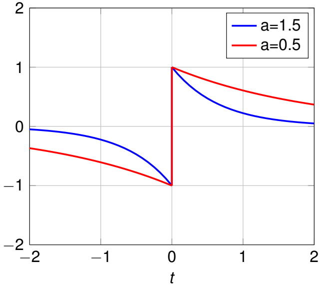


Fourier Transform

$$\mathrm{sgn}(t) \rightleftharpoons \frac{1}{j\pi f}$$

Signum Function

$$g(t) = \begin{cases} e^{-at}, & t > 0 \\ 0, & t = 0 \\ -e^{at}, & t < 0 \end{cases}$$

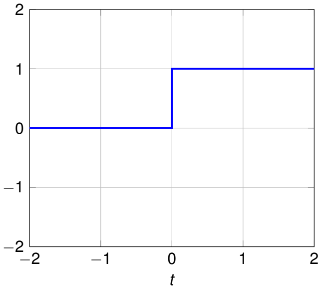


$$\text{sgn}(t) = \lim_{a \to 0^+} g(t)$$

$$G(f) = \frac{-j4\pi f}{a^2 + (2\pi f)^2}$$

Unit Step Function

$$u(t) = \begin{cases} 1, & t > 0 \\ \frac{1}{2}, & t = 0 \\ 0, & t < 0 \end{cases}$$



Fourier Transform

$$u(t) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$$

$$u(t) = \frac{1}{2}[\mathrm{sgn}(t) + 1]$$

Properties of Fourier Transform

- Differentiation

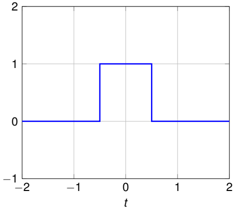
$$\frac{d}{dt}x(t) \rightleftharpoons j2\pi f\ X(f)$$

- Integration

$$\int_{-\infty}^{t} x(\tau)\ d\tau \rightleftharpoons \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

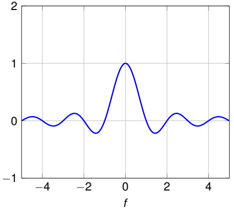
Rectangular Pulse

$$\Pi(t) = \begin{cases} 1, & |t| \leq \frac{1}{2} \\ 0, & |t| > \frac{1}{2} \end{cases}$$



Fourier Transform

$$\Pi\left(\frac{t}{T}\right) \rightleftharpoons T\text{sinc}(fT)$$



Thanks for your attention