Gaussian Random Vectors and Processes

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Gaussian Random Vectors
Jointly Gaussian Random Variables

Definition (Jointly Gaussian RVs)
Random variables $X_1, X_2, \ldots, X_n$ are jointly Gaussian if any non-trivial linear combination is a Gaussian random variable.

$$a_1 X_1 + \cdots + a_n X_n \text{ is Gaussian for all } (a_1, \ldots, a_n) \in \mathbb{R}^n \setminus 0$$

Example (Not Jointly Gaussian)
$X \sim N(0, 1)$

$$Y = \begin{cases} X, & \text{if } |X| > 1 \\ -X, & \text{if } |X| \leq 1 \end{cases}$$

$Y \sim N(0, 1)$ and $X + Y$ is not Gaussian.
Gaussian Random Vector

Definition (Gaussian Random Vector)
A random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ whose components are jointly Gaussian.

Notation
$\mathbf{X} \sim N(\mathbf{m}, \mathbf{C})$ where

$$
\begin{align*}
\mathbf{m} &= E[\mathbf{X}] \text{ is the } n \times 1 \text{ mean vector} \\
\mathbf{C} &= E[(\mathbf{X} - \mathbf{m})(\mathbf{X} - \mathbf{m})^T] \text{ is the } n \times n \text{ covariance matrix}
\end{align*}
$$

$m_i = E[X_i], C_{ij} = E[(X_i - m_i)(X_j - m_j)] = \text{cov}(X_i, X_j)$

Definition (Joint Gaussian Density)
For a Gaussian random vector, $\mathbf{C}$ is invertible and the joint density is given by

$$
\rho(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp \left( -\frac{1}{2} (\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m}) \right)
$$

For derivation, see Problems 3.31(f) and 3.32 in Madhow’s book.
Uncorrelated Jointly Gaussian RVs are Independent

If $X_1, \ldots, X_n$ are jointly Gaussian and pairwise uncorrelated, then they are independent. For pairwise uncorrelated random variables,

$$C_{ij} = E[(X_i - m_i)(X_j - m_j)] = \begin{cases} 0 & \text{if } i \neq j \\ \sigma_i^2 & \text{otherwise.} \end{cases}$$

The joint probability density function is given by

$$\rho(x) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp \left( -\frac{1}{2} (x - m)^T C^{-1} (x - m) \right)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp \left( - \frac{(x_i - m_i)^2}{2\sigma_i^2} \right)$$

where $m_i = E[X_i]$ and $\sigma_i^2 = \text{var}(X_i)$. 
Uncorrelated Gaussian RVs may not be Independent

Example

- $X \sim N(0, 1)$
- $W$ is equally likely to be +1 or -1
- $W$ is independent of $X$
- $Y = WX$
- $Y \sim N(0, 1)$
- $X$ and $Y$ are uncorrelated
- $X$ and $Y$ are not independent
Gaussian Random Processes
**Gaussian Random Process**

**Definition**
A random process $X(t)$ is Gaussian if its samples $X(t_1), \ldots, X(t_n)$ are jointly Gaussian for any $n \in \mathbb{N}$ and distinct sample locations $t_1, t_2, \ldots, t_n$.

Let $X = [X(t_1) \cdots X(t_n)]^T$ be the vector of samples. The joint density is given by

$$p(x) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp \left( -\frac{1}{2} (x - m)^T C^{-1} (x - m) \right)$$

where

$$m = E[X], \quad C = E \left[(X - m)(X - m)^T\right]$$
Properties of Gaussian Random Process

- The mean and autocorrelation functions completely characterize a Gaussian random process.
- Wide-sense stationary Gaussian processes are strictly stationary.
- If the input to a stable linear filter is a Gaussian random process, the output is also a Gaussian random process.

\[ X(t) \xrightarrow{h(t)} Y(t) \]
White Gaussian Noise

Definition
A zero mean WSS Gaussian random process with power spectral density

\[ S_n(f) = \frac{N_0}{2}. \]

\[ \frac{N_0}{2} \] is termed the two-sided PSD and has units Watts per Hertz.

Remarks

- Autocorrelation function \( R_n(\tau) = \frac{N_0}{2} \delta(\tau) \)
- **Infinite Power!** Ideal model of Gaussian noise occupying more bandwidth than the signals of interest.
White Gaussian Noise through Correlators

- Consider the output of a correlator with WGN input
  \[ Z = \int_{-\infty}^{\infty} n(t)u(t) \, dt = \langle n, u \rangle \]

  where \( u(t) \) is a deterministic finite-energy signal

- \( Z \) is a Gaussian random variable

- The mean of \( Z \) is
  \[ E[Z] = \int_{-\infty}^{\infty} E[n(t)] \, u(t) \, dt = 0 \]

- The variance of \( Z \) is
  \[ \text{var}(Z) = E[(\langle n, u \rangle)^2] = E\left[ \int n(t)u(t) \, dt \int n(s)u(s) \, ds \right] \]
  \[ = \int \int u(t)u(s) E[n(t)n(s)] \, dt \, ds \]
  \[ = \int \int u(t)u(s) \frac{N_0}{2} \delta(t - s) \, dt \, ds \]
  \[ = \frac{N_0}{2} \int u^2(t) \, dt = \frac{N_0}{2} ||u||^2 \]
White Gaussian Noise through Correlators

Proposition
Let $u_1(t)$ and $u_2(t)$ be linearly independent finite-energy signals and let $n(t)$ be WGN with PSD $S_n(f) = \frac{N_0}{2}$. Then $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian with covariance

$$\text{cov} (\langle n, u_1 \rangle, \langle n, u_2 \rangle) = \frac{N_0}{2} \langle u_1, u_2 \rangle.$$

Proof
To prove that $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are jointly Gaussian, consider a non-trivial linear combination $a\langle n, u_1 \rangle + b\langle n, u_2 \rangle$

$$a\langle n, u_1 \rangle + b\langle n, u_2 \rangle = \int n(t) [au_1(t) + bu_2(t)] \, dt.$$

This is the result of passing $n(t)$ through a correlator. So it is a Gaussian random variable.
White Gaussian Noise through Correlators

Proof (continued)

\[
\text{cov}(\langle n, u_1 \rangle, \langle n, u_2 \rangle) = E[\langle n, u_1 \rangle \langle n, u_2 \rangle] \\
= E \left[ \int n(t)u_1(t) \, dt \int n(s)u_2(s) \, ds \right] \\
= \int \int u_1(t)u_2(s)E[n(t)n(s)] \, dt \, ds \\
= \int \int u_1(t)u_2(s)N_0 \frac{1}{2} \delta(t - s) \, dt \, ds \\
= \frac{N_0}{2} \int u_1(t)u_2(t) \, dt \\
= \frac{N_0}{2} \langle u_1, u_2 \rangle
\]

If \( u_1(t) \) and \( u_2(t) \) are orthogonal, \( \langle n, u_1 \rangle \) and \( \langle n, u_2 \rangle \) are independent.
Thanks for your attention