Optimal Receiver for the AWGN Channel

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Additive White Gaussian Noise Channel

$$s(t) \longrightarrow \boxed{\begin{array}{c}\text{AWGN}\\\text{Channel}\end{array}} \longrightarrow y(t)$$

$$y(t) = s(t) + n(t)$$

$s(t)$ Transmitted Signal

$y(t)$ Received Signal

$n(t)$ White Gaussian Noise

$$S_n(f) = \frac{N_0}{2} = \sigma^2$$

$$R_n(\tau) = \sigma^2 \delta(\tau)$$

M-ary Signaling in AWGN Channel

- One of M continuous-time signals $s_1(t), \ldots, s_M(t)$ is sent
- The received signal is the transmitted signal corrupted by AWGN
- M hypotheses with prior probabilities π_i, $i = 1, \ldots, M$

$$
\begin{aligned}
H_1 &: \quad y(t) = s_1(t) + n(t) \\
H_2 &: \quad y(t) = s_2(t) + n(t) \\
&\vdots \qquad\qquad \vdots \\
H_M &: \quad y(t) = s_M(t) + n(t)
\end{aligned}
$$

- Random variables are easier to handle than random processes
- We derive an equivalent M-ary hypothesis testing problem involving only random vectors

Restriction to Signal Space is Optimal

Theorem

For the M-ary hypothesis testing given by

$$
\begin{aligned}
H_1 &: \quad y(t) = s_1(t) + n(t) \\
\vdots & \qquad\qquad \vdots \\
H_M &: \quad y(t) = s_M(t) + n(t)
\end{aligned}
$$

there is no loss in detection performance by using the optimal decision rule for the following M-ary hypothesis testing problem

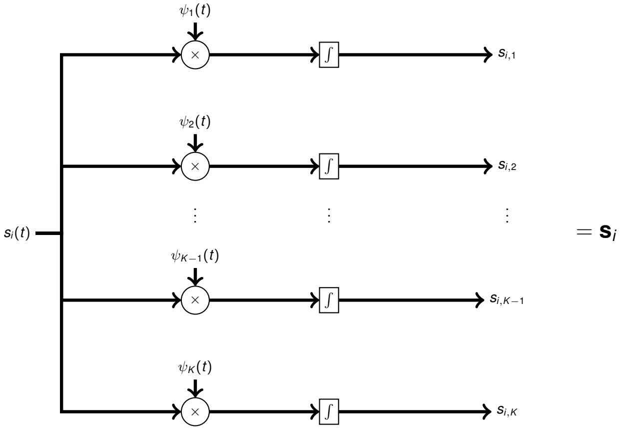
$$
\begin{aligned}
H_1 &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\
\vdots & \qquad\quad \vdots \\
H_M &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N}
\end{aligned}
$$

where $\mathbf{Y}$, $\mathbf{s}_i$ *and* $\mathbf{N}$ *are the projections of* $y(t)$, $s_i(t)$ *and* $n(t)$ *respectively onto the signal space spanned by* $\{s_i(t)\}$.

Projection of Signals onto Signal Space

- Consider an orthonormal basis $\{\psi_i(t) | i = 1, \ldots, K\}$ for the space spanned by $\{s_i(t) | i = 1, \ldots, M\}$
- Projection of $s_i(t)$ onto the signal space is

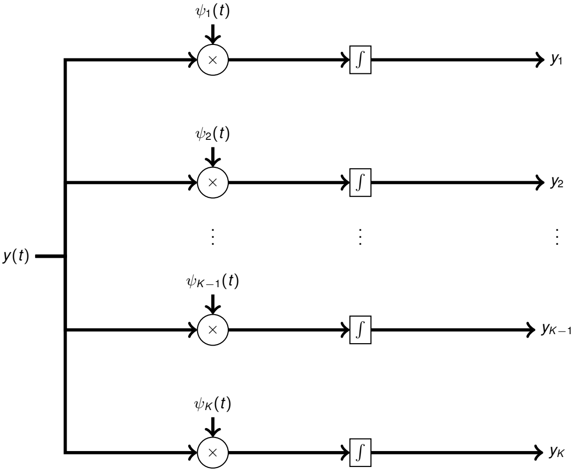
$$\mathbf{s}_i = \left[\langle s_i, \psi_1 \rangle \quad \cdots \quad \langle s_i, \psi_K \rangle\right]^T$$



Projection of Observed Signal onto Signal Space

- Projection of $y(t)$ onto the signal space is

$$\mathbf{Y} = \begin{bmatrix} \langle y, \psi_1 \rangle & \cdots & \langle y, \psi_K \rangle \end{bmatrix}^T$$

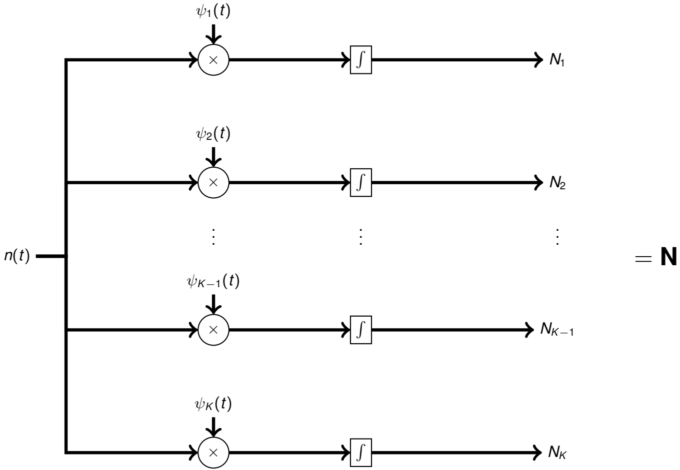


$= \mathbf{y}$

Projection of Noise onto Signal Space

- Projection of $n(t)$ onto the signal space is

$$\mathbf{N} = \begin{bmatrix} \langle n, \psi_1 \rangle & \cdots & \langle n, \psi_K \rangle \end{bmatrix}^T$$



Proof of Theorem

- $\mathbf{Y} = \left[\langle y, \psi_1 \rangle \quad \cdots \quad \langle y, \psi_K \rangle\right]^T$
- Component of $y(t)$ orthogonal to the signal space is

$$y^{\perp}(t) = y(t) - \sum_{i=1}^{K} \langle y, \psi_i \rangle \psi_i(t)$$

- $y(t)$ is equivalent to $(\mathbf{Y}, y^{\perp}(t))$
- We claim that $y^{\perp}(t)$ is an irrelevant statistic

$$\begin{aligned} y^{\perp}(t) &= y(t) - \sum_{i=1}^{K} \langle y, \psi_i \rangle \psi_i(t) \\ &= s_i(t) + n(t) - \sum_{j=1}^{K} \langle s_i + n, \psi_j \rangle \psi_j(t) \\ &= n(t) - \sum_{j=1}^{K} \langle n, \psi_j \rangle \psi_j(t) = n^{\perp}(t) \end{aligned}$$

  where $n^{\perp}(t)$ is the component of $n(t)$ orthogonal to the signal space.
- $n^{\perp}(t)$ is independent of which $s_i(t)$ was transmitted which makes $y^{\perp}(t)$ an irrelevant statistic.

M-ary Signaling in AWGN Channel

- M hypotheses with prior probabilities π_i, $i = 1, \dots, M$

$$\begin{aligned} H_1 \quad &: \quad \mathbf{Y} = \mathbf{s}_1 + \mathbf{N} \\ \vdots \quad & \qquad\quad \vdots \\ H_M \quad &: \quad \mathbf{Y} = \mathbf{s}_M + \mathbf{N} \end{aligned}$$

$$\begin{aligned} \mathbf{Y} &= \begin{bmatrix} \langle y, \psi_1 \rangle & \cdots & \langle y, \psi_K \rangle \end{bmatrix}^T \\ \mathbf{s}_i &= \begin{bmatrix} \langle s_i, \psi_1 \rangle & \cdots & \langle s_i, \psi_K \rangle \end{bmatrix}^T \\ \mathbf{N} &= \begin{bmatrix} \langle n, \psi_1 \rangle & \cdots & \langle n, \psi_K \rangle \end{bmatrix}^T \end{aligned}$$

- $\mathbf{N} \sim N(\mathbf{m}, \mathbf{C})$ where $\mathbf{m} = \mathbf{0}$ and $\mathbf{C} = \sigma^2 \mathbf{I}$

$$\text{cov}\left(\langle n, \psi_1 \rangle, \langle n, \psi_2 \rangle\right) = \sigma^2 \langle \psi_1, \psi_2 \rangle.$$

Optimal Receiver for the AWGN Channel

Theorem (MPE Decision Rule)

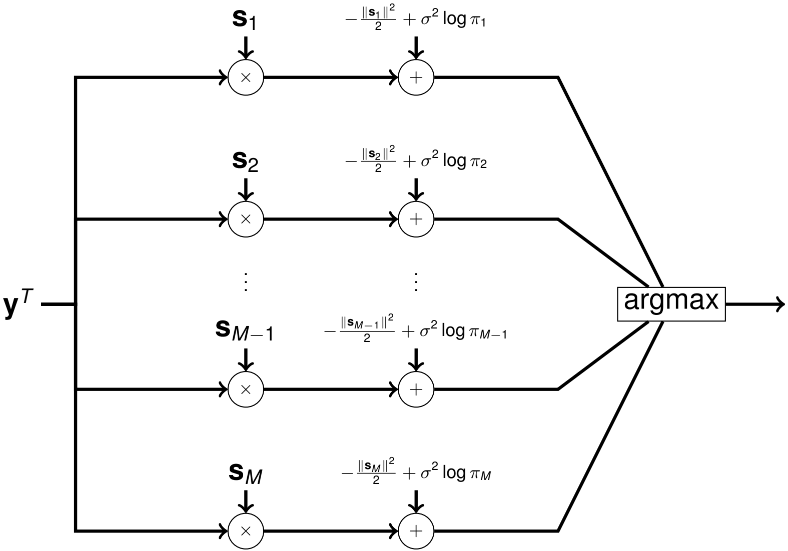
The MPE decision rule for M-ary signaling in AWGN channel is given by

$$\begin{aligned}\delta_{MPE}(\mathbf{y}) &= \underset{1\le i\le M}{\operatorname{argmin}} \|\mathbf{y}-\mathbf{s}_i\|^2 - 2\sigma^2 \log \pi_i \\ &= \underset{1\le i\le M}{\operatorname{argmax}} \langle \mathbf{y}, \mathbf{s}_i\rangle - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i\end{aligned}$$

Proof

$$\begin{aligned}\delta_{MPE}(\mathbf{y}) &= \underset{1\le i\le M}{\operatorname{argmax}}\, \pi_i p_i(\mathbf{y}) \\ &= \underset{1\le i\le M}{\operatorname{argmax}}\, \pi_i \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)\end{aligned}$$

MPE Decision Rule



Continuous-Time Version of MPE Rule

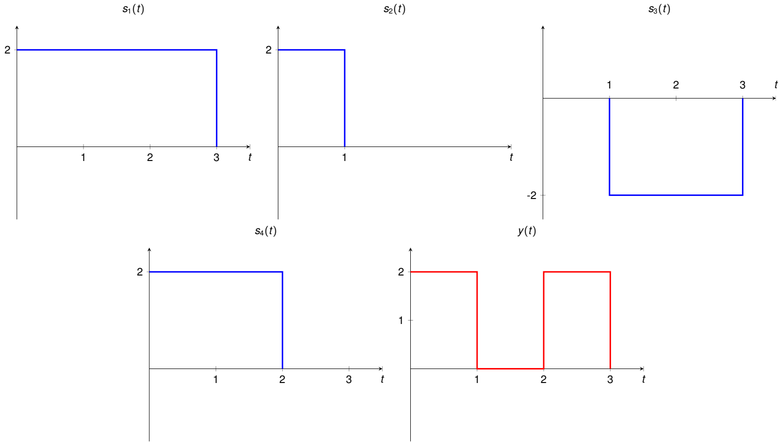
- Discrete-time version

$$\delta_{MPE}(\mathbf{y}) = \underset{1 \le i \le M}{\operatorname{argmax}} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2} + \sigma^2 \log \pi_i$$

- Continuous-time version

$$\delta_{MPE}(y) = \underset{1 \le i \le M}{\operatorname{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2} + \sigma^2 \log \pi_i$$

MPE Decision Rule Example



Let $\pi_1 = \pi_2 = \dfrac{1}{3}, \pi_3 = \pi_4 = \dfrac{1}{6}, \sigma^2 = 1,$ and $\log 2 = 0.69$.

ML Receiver for the AWGN Channel

Theorem (ML Decision Rule)

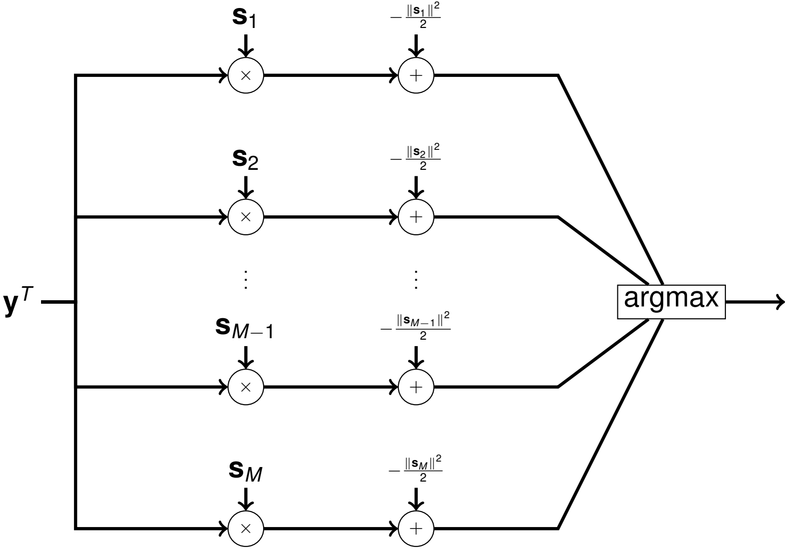
The ML decision rule for M-ary signaling in AWGN channel is given by

$$\begin{aligned}\delta_{ML}(\mathbf{y}) &= \underset{1\le i\le M}{\operatorname{argmin}} \|\mathbf{y}-\mathbf{s}_i\|^2 \\ &= \underset{1\le i\le M}{\operatorname{argmax}} \langle \mathbf{y}, \mathbf{s}_i\rangle - \frac{\|\mathbf{s}_i\|^2}{2}\end{aligned}$$

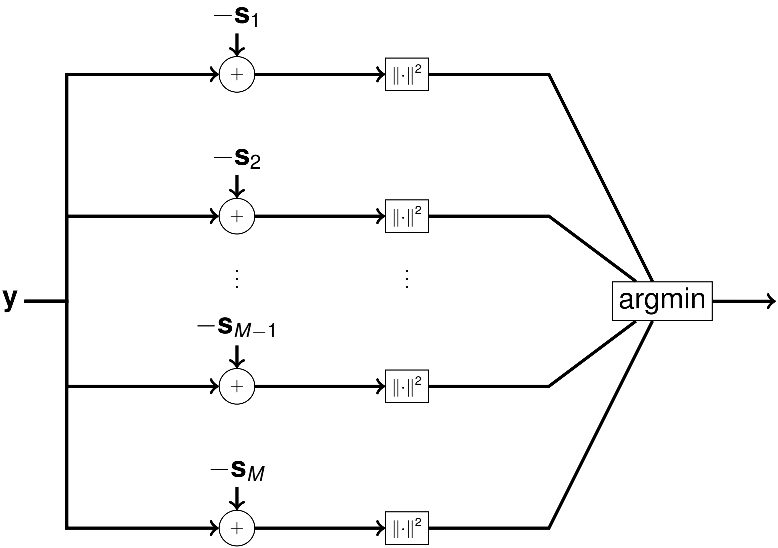
Proof

$$\begin{aligned}\delta_{ML}(\mathbf{y}) &= \underset{1\le i\le M}{\operatorname{argmax}}\, p_i(\mathbf{y}) \\ &= \underset{1\le i\le M}{\operatorname{argmax}} \exp\left(-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right)\end{aligned}$$

ML Decision Rule



ML Decision Rule



Continuous-Time Version of ML Rule

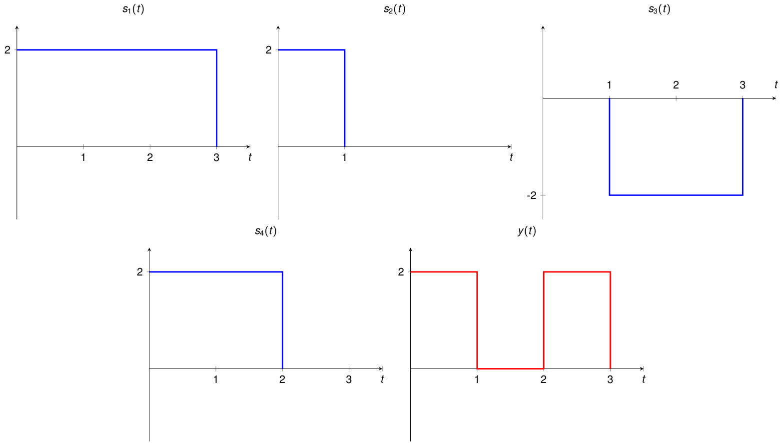
- Discrete-time version

$$\delta_{ML}(\mathbf{y}) = \operatorname*{argmax}_{1 \le i \le M} \langle \mathbf{y}, \mathbf{s}_i \rangle - \frac{\|\mathbf{s}_i\|^2}{2}$$

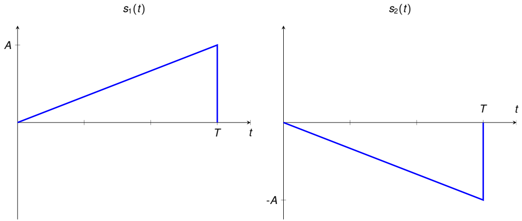
- Continuous-time version

$$\delta_{ML}(y) = \operatorname*{argmax}_{1 \le i \le M} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$$

ML Decision Rule Example



ML Decision Rule for Antipodal Signaling



$$\delta_{ML}(y) = \underset{1\le i\le 2}{\text{argmax}} \langle y, s_i\rangle - \frac{\|s_i\|^2}{2} = \underset{1\le i\le 2}{\text{argmax}} \langle y, s_i\rangle$$

$$\delta_{ML}(y) = 1 \iff \langle y, s_1\rangle \ge \langle y, s_2\rangle \iff \langle y, s_1\rangle \ge 0$$

$$\langle y, s_1\rangle = \int_0^T y(\tau)s_1(\tau)\, d\tau = (y \star s_{MF})(T)$$

where $s_{MF}(t) = s_1(T - t)$ is the matched filter.

Thanks for your attention