Power Spectral Density of Digitally Modulated Signals

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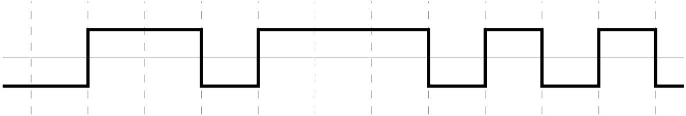
PSD Definition for Digitally Modulated Signals

- Consider a real binary PAM signal

$$u(t) = \sum_{n=-\infty}^{\infty} b_n g(t - nT)$$

where $b_n = \pm 1$ with equal probability and $g(t)$ is a baseband pulse of duration T



- ~~PSD~~ $\cancel{= \mathcal{F}[R_u(\tau)]}$ Neither SSS nor WSS

Cyclostationary Random Process

Definition (Cyclostationary RP)

A random process $X(t)$ is cyclostationary with respect to time interval T if it is statistically indistinguishable from $X(t - kT)$ for any integer k.

Definition (Wide Sense Cyclostationary RP)

A random process $X(t)$ is wide sense cyclostationary with respect to time interval T if the mean and autocorrelation functions satisfy

$$\begin{aligned} m_X(t) &= m_X(t - T) \quad \text{for all } t, \\ R_X(t_1, t_2) &= R_X(t_1 - T, t_2 - T) \quad \text{for all } t_1, t_2. \end{aligned}$$

Power Spectral Density of a Cyclostationary Process

To obtain the PSD of a cyclostationary process with period T

- Calculate autocorrelation of cyclostationary process $R_X(t, t-\tau)$
- Average autocorrelation between 0 and T, $R_X(\tau) = \frac{1}{T}\int_0^T R_X(t, t-\tau)\, dt$
- Calculate Fourier transform of averaged autocorrelation $R_X(\tau)$

Power Spectral Density of a Realization

Time windowed realizations have finite energy

$$
\begin{aligned}
x_{T_o}(t) &= x(t) I_{[-\frac{T_o}{2}, \frac{T_o}{2}]}(t) \\
S_{T_o}(f) &= \mathcal{F}(x_{T_o}(t)) \\
\hat{S}_x(f) &= \frac{|S_{T_o}(f)|^2}{T_o} \quad \text{(PSD Estimate)}
\end{aligned}
$$

PSD of a realization

$$
\bar{S}_x(f) = \lim_{T_o \to \infty} \frac{|S_{T_o}(f)|^2}{T_o}
$$

$$
\frac{|S_{T_o}(f)|^2}{T_o} \rightleftharpoons \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x_{T_o}(u) x_{T_o}^*(u - \tau)\, du = \hat{R}_x(\tau)
$$

Power Spectral Density of a Cyclostationary Process

$X(t)X^*(t-\tau) \sim X(t+T)X^*(t+T-\tau)$ for cyclostationary $X(t)$

$$
\begin{aligned}
\hat{R}_x(\tau) &= \frac{1}{T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t-\tau)\,dt \\
&= \frac{1}{KT}\int_{-\frac{KT}{2}}^{\frac{KT}{2}} x(t)x^*(t-\tau)\,dt \qquad \text{for } T_o = KT \\
&= \frac{1}{T}\int_0^T \frac{1}{K}\sum_{k=-\frac{K}{2}}^{\frac{K}{2}} x(t+kT)x^*(t+kT-\tau)\,dt \\
&\xrightarrow[K\to\infty]{} \frac{1}{T}\int_0^T E\left[X(t)X^*(t-\tau)\right]\,dt \\
&= \frac{1}{T}\int_0^T R_X(t,t-\tau)\,dt = R_X(\tau)
\end{aligned}
$$

PSD of a cyclostationary process = $\mathcal{F}[R_X(\tau)]$

PSD of a Linearly Modulated Signal

- Consider

$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$

- $u(t)$ is cyclostationary wrt to T if $\{b_n\}$ is stationary
- $u(t)$ is wide sense cyclostationary wrt to T if $\{b_n\}$ is WSS
- Suppose $R_b[k] = E[b_n b_{n-k}^*]$
- Let $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$
- The PSD of $u(t)$ is given by

$$S_u(f) = S_b\left(e^{j2\pi fT}\right) \frac{|P(f)|^2}{T}$$

PSD of a Linearly Modulated Signal

$$
\begin{aligned}
R_u(\tau) & \\
&= \frac{1}{T}\int_0^T R_u(t+\tau, t)\, dt \\
&= \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E\left[b_n b_m^* p(t - nT + \tau) p^*(t - mT)\right]\, dt \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-mT}^{-(m-1)T} E\left[b_{m+k} b_m^* p(\lambda - kT + \tau) p^*(\lambda)\right]\, d\lambda \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} E\left[b_{m+k} b_m^* p(\lambda - kT + \tau) p^*(\lambda)\right]\, d\lambda \\
&= \frac{1}{T} \sum_{k=-\infty}^{\infty} R_b[k] \int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda)\, d\lambda
\end{aligned}
$$

PSD of a Linearly Modulated Signal

$$R_u(\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} R_b[k] \int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda)\, d\lambda$$

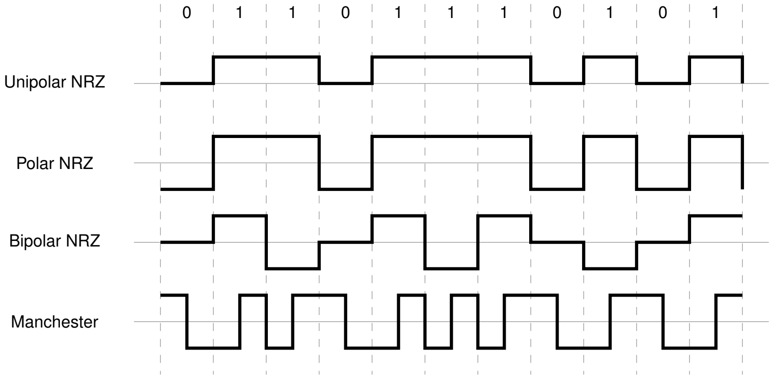
$$\begin{aligned}
\int_{-\infty}^{\infty} p(\lambda + \tau) p^*(\lambda)\, d\lambda \quad &\rightleftharpoons \quad |P(f)|^2 \\
\int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda)\, d\lambda \quad &\rightleftharpoons \quad |P(f)|^2 e^{-j2\pi f kT}
\end{aligned}$$

$$\begin{aligned}
S_u(f) = \mathcal{F}\left[R_u(\tau)\right] \quad &= \quad \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi f kT} \\
&= \quad S_b\left(e^{j2\pi f T}\right) \frac{|P(f)|^2}{T}
\end{aligned}$$

where $S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k}$.

Power Spectral Density of Line Codes

Line Codes



Further reading: *Digital Communications*, Simon Haykin, Chapter 6

Unipolar NRZ

- Symbols independent and equally likely to be 0 or A

$$P\left(b[n]=0\right)=P\left(b[n]=A\right)=\frac{1}{2}$$

- Autocorrelation of $b[n]$ sequence

$$R_b[k]=\begin{cases}\frac{A^2}{2} & k=0\\ \frac{A^2}{4} & k\neq 0\end{cases}$$

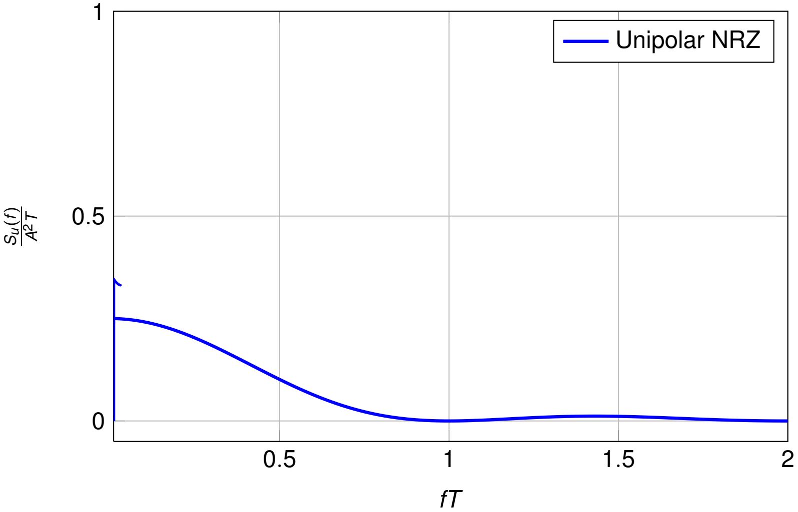
- $p(t)=I_{[0,T)}(t)\Rightarrow P(f)=T\text{sinc}(fT)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f)=\frac{|P(f)|^2}{T}\sum_{k=-\infty}^{\infty}R_b[k]e^{-j2\pi kfT}$$

Unipolar NRZ

$$
\begin{aligned}
S_u(f) &= \frac{A^2 T}{4}\mathrm{sinc}^2(fT) + \frac{A^2 T}{4}\mathrm{sinc}^2(fT) \sum_{k=-\infty}^{\infty} e^{-j2\pi kfT} \\
&= \frac{A^2 T}{4}\mathrm{sinc}^2(fT) + \frac{A^2}{4}\mathrm{sinc}^2(fT) \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \\
&= \frac{A^2 T}{4}\mathrm{sinc}^2(fT) + \frac{A^2}{4}\delta(f)
\end{aligned}
$$

Normalized PSD plot



Polar NRZ

- Symbols independent and equally likely to be $-A$ or A

$$P\left(b[n] = -A\right) = P\left(b[n] = A\right) = \frac{1}{2}$$

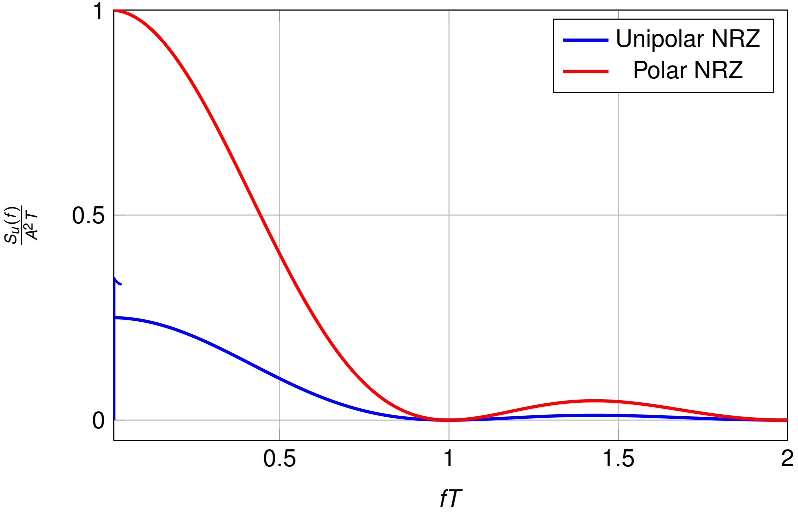
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

- $P(f) = T\text{sinc}(fT)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f) = A^2 T\text{sinc}^2(fT)$$

Normalized PSD plots



Manchester

- Symbols independent and equally likely to be $-A$ or A

$$P\left(b[n] = -A\right) = P\left(b[n] = A\right) = \frac{1}{2}$$

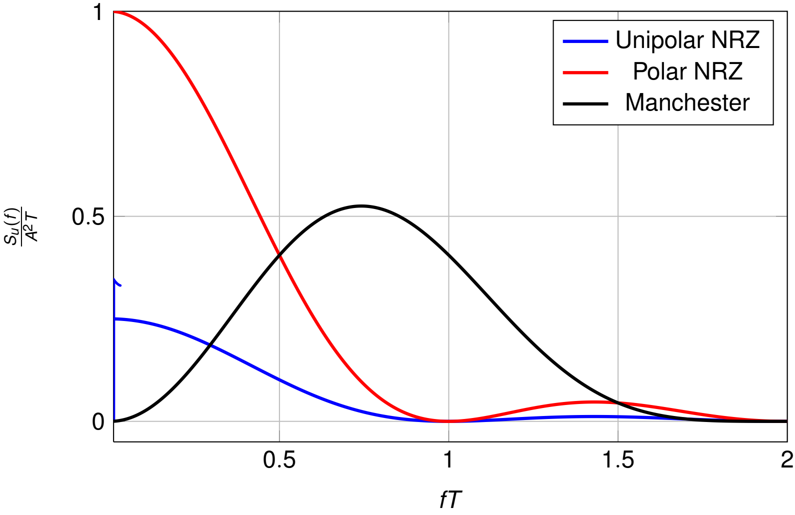
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

- $P(f) = jT\text{sinc}\left(\frac{fT}{2}\right)\sin\left(\frac{\pi fT}{2}\right)e^{-j\pi fT}$
- Power Spectral Density

$$S_u(f) = A^2 T\text{sinc}^2\left(\frac{fT}{2}\right)\sin^2\left(\frac{\pi fT}{2}\right)$$

Normalized PSD plots



Bipolar NRZ

- Successive 1's have alternating polarity

$$\begin{aligned} 0 &\rightarrow \text{Zero amplitude} \\ 1 &\rightarrow +A \text{ or } -A \end{aligned}$$

- Probability mass function of $b[n]$

$$\begin{aligned} P\left(b[n]=0\right) &= \frac{1}{2} \\ P\left(b[n]=-A\right) &= \frac{1}{4} \\ P\left(b[n]=A\right) &= \frac{1}{4} \end{aligned}$$

- Symbols are identically distributed but they are not independent

Bipolar NRZ

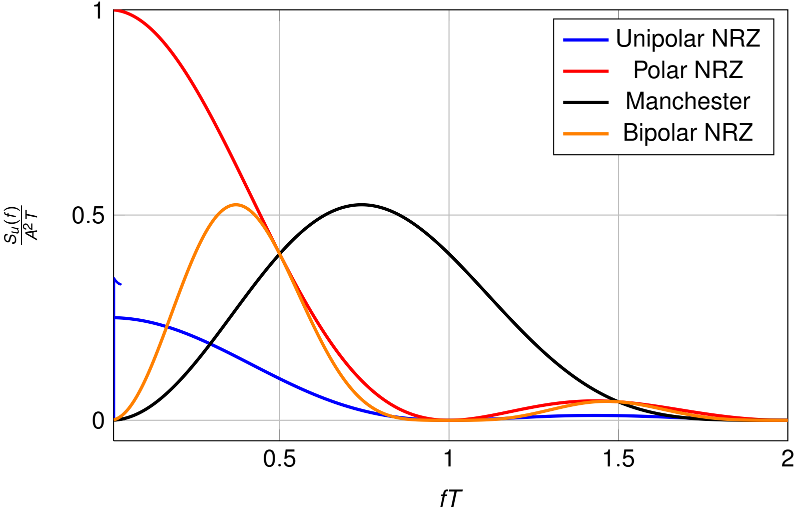
- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} A^2/2 & k = 0 \\ -A^2/4 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

- Power Spectral Density

$$\begin{aligned} S_u(f) &= T\text{sinc}^2(fT)\left[\frac{A^2}{2} - \frac{A^2}{4}\left(e^{j2\pi fT} + e^{-j2\pi fT}\right)\right] \\ &= \frac{A^2 T}{2}\text{sinc}^2(fT)\left[1 - \cos(2\pi fT)\right] \\ &= A^2 T\text{sinc}^2(fT)\sin^2(\pi fT) \end{aligned}$$

Normalized PSD plots



Thanks for your attention