Power Spectral Density of Digitally Modulated Signals

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PSD Definition for Digitally Modulated Signals

- Consider a real binary PAM signal

\[ u(t) = \sum_{n=\infty}^{\infty} b_n g(t - nT) \]

where \( b_n = \pm 1 \) with equal probability and \( g(t) \) is a baseband pulse of duration \( T \)

- PSD \( = \mathcal{F}[R_u(\tau)] \) Neither SSS nor WSS
Cyclostationary Random Process

Definition (Cyclostationary RP)
A random process $X(t)$ is cyclostationary with respect to time interval $T$ if it is statistically indistinguishable from $X(t - kT)$ for any integer $k$.

Definition (Wide Sense Cyclostationary RP)
A random process $X(t)$ is wide sense cyclostationary with respect to time interval $T$ if the mean and autocorrelation functions satisfy

$$m_X(t) = m_X(t - T) \quad \text{for all } t,$$

$$R_X(t_1, t_2) = R_X(t_1 - T, t_2 - T) \quad \text{for all } t_1, t_2.$$
Power Spectral Density of a Cyclostationary Process

To obtain the PSD of a cyclostationary process with period $T$

- Calculate autocorrelation of cyclostationary process $R_X(t, t - \tau)$
- Average autocorrelation between 0 and $T$, $R_X(\tau) = \frac{1}{T} \int_{0}^{T} R_X(t, t - \tau) \, dt$
- Calculate Fourier transform of averaged autocorrelation $R_X(\tau)$
Power Spectral Density of a Realization

Time windowed realizations have finite energy

\[ x_{T_0}(t) = x(t) \mathcal{I}_{[-\frac{T_0}{2}, \frac{T_0}{2}]}(t) \]

\[ S_{T_0}(f) = \mathcal{F}(x_{T_0}(t)) \]

\[ \hat{S}_x(f) = \frac{|S_{T_0}(f)|^2}{T_0} \quad \text{(PSD Estimate)} \]

PSD of a realization

\[ \bar{S}_x(f) = \lim_{T_0 \to \infty} \frac{|S_{T_0}(f)|^2}{T_0} \]

\[ \frac{|S_{T_0}(f)|^2}{T_0} \Rightarrow \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x_{T_0}(u)x_{T_0}^*(u - \tau) \, du = \hat{R}_x(\tau) \]
Power Spectral Density of a Cyclostationary Process

\[ X(t)X^*(t - \tau) \sim X(t + T)X^*(t + T - \tau) \] for cyclostationary \( X(t) \)

\[
\hat{R}_X(\tau) = \frac{1}{T_o} \int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} x(t)x^*(t - \tau) \, dt
\]

\[
= \frac{1}{KT} \int_{-\frac{KT}{2}}^{\frac{KT}{2}} x(t)x^*(t - \tau) \, dt \quad \text{for} \ T_o = KT
\]

\[
= \frac{1}{T} \int_0^T \frac{1}{K} \sum_{k=-\frac{K}{2}}^{\frac{K}{2}} x(t + kT)x^*(t + kT - \tau) \, dt
\]

\[
\xrightarrow{K \to \infty} \frac{1}{T} \int_0^T E[X(t)X^*(t - \tau)] \, dt
\]

\[
= \frac{1}{T} \int_0^T R_X(t, t - \tau) \, dt = R_X(\tau)
\]

PSD of a cyclostationary process = \( \mathcal{F}[R_X(\tau)] \)
**PSD of a Linearly Modulated Signal**

- Consider
  \[ u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT) \]

- \( u(t) \) is cyclostationary wrt to \( T \) if \( \{b_n\} \) is stationary
- \( u(t) \) is wide sense cyclostationary wrt to \( T \) if \( \{b_n\} \) is WSS
- Suppose \( R_b[k] = E[b_n b_{n-k}^*] \)
- Let \( S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k] z^{-k} \)
- The PSD of \( u(t) \) is given by
  \[ S_u(f) = S_b \left( e^{j2\pi f T} \right) \frac{|P(f)|^2}{T} \]
PSD of a Linearly Modulated Signal

\[ R_u(\tau) = \frac{1}{T} \int_0^T R_u(t + \tau, t) \, dt \]

\[ = \frac{1}{T} \int_0^T \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[b_n b_m^* p(t - nT + \tau) p^*(t - mT)] \, dt \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-(m-1)T}^{-(m-1)T} E[b_{m+k} b_m^* p(\lambda - kT + \tau) p^*(\lambda)] \, d\lambda \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} E[b_{m+k} b_m^* p(\lambda - kT + \tau) p^*(\lambda)] \, d\lambda \]

\[ = \frac{1}{T} \sum_{k=-\infty}^{\infty} R_b[k] \int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda) \, d\lambda \]
PSD of a Linearly Modulated Signal

\[ R_u(\tau) = \frac{1}{T} \sum_{k=-\infty}^{\infty} R_b[k] \int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda) \, d\lambda \]

\[ \int_{-\infty}^{\infty} p(\lambda + \tau) p^*(\lambda) \, d\lambda \quad \Rightarrow \quad |P(f)|^2 \]

\[ \int_{-\infty}^{\infty} p(\lambda - kT + \tau) p^*(\lambda) \, d\lambda \quad \Rightarrow \quad |P(f)|^2 e^{-j2\pi fkT} \]

\[ S_u(f) = \mathcal{F}[R_u(\tau)] = \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi fkT} \]

\[ = S_b \left( e^{j2\pi fT} \right) \frac{|P(f)|^2}{T} \]

where \( S_b(z) = \sum_{k=-\infty}^{\infty} R_b[k]z^{-k} \).
Power Spectral Density of Line Codes
Line Codes

Further reading: *Digital Communications*, Simon Haykin, Chapter 6
Unipolar NRZ

- Symbols independent and equally likely to be 0 or $A$

$$P(b[n] = 0) = P(b[n] = A) = \frac{1}{2}$$

- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} 
\frac{A^2}{2} & k = 0 \\
\frac{A^2}{4} & k \neq 0
\end{cases}$$

- $p(t) = l_{[0,T]}(t) \Rightarrow P(f) = T \text{sinc}(fT) e^{-j\pi fT}$

- Power Spectral Density

$$S_u(f) = \frac{|P(f)|^2}{T} \sum_{k=-\infty}^{\infty} R_b[k] e^{-j2\pi kfT}$$
Unipolar NRZ

\[ S_u(f) = \frac{A^2 T}{4} \text{sinc}^2(fT) + \frac{A^2 T}{4} \text{sinc}^2(fT) \sum_{k=-\infty}^{\infty} e^{-j2\pi kfT} \]

\[ = \frac{A^2 T}{4} \text{sinc}^2(fT) + \frac{A^2}{4} \text{sinc}^2(fT) \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right) \]

\[ = \frac{A^2 T}{4} \text{sinc}^2(fT) + \frac{A^2}{4} \delta(f) \]
Normalized PSD plot

\[ \frac{S_u(f)}{A^2 T} \]

\[ fT \]

Unipolar NRZ
Polar NRZ

- Symbols independent and equally likely to be $-A$ or $A$

$$P(b[n] = -A) = P(b[n] = A) = \frac{1}{2}$$

- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} 
A^2 & k = 0 \\
0 & k \neq 0 
\end{cases}$$

- $P(f) = T \text{sinc}(fT) e^{-j\pi fT}$

- Power Spectral Density

$$S_u(f) = A^2 T \text{sinc}^2(fT)$$
Normalized PSD plots
Manchester

- Symbols independent and equally likely to be $-A$ or $A$

\[ P(b[n] = -A) = P(b[n] = A) = \frac{1}{2} \]

- Autocorrelation of $b[n]$ sequence

\[ R_b[k] = \begin{cases} 
A^2 & k = 0 \\
0 & k \neq 0 
\end{cases} \]

- $P(f) = jT \text{sinc} \left( \frac{fT}{2} \right) \sin \left( \frac{\pi fT}{2} \right) e^{-j\pi fT}$

- Power Spectral Density

\[ S_u(f) = A^2 T \text{sinc}^2 \left( \frac{fT}{2} \right) \sin^2 \left( \frac{\pi fT}{2} \right) \]
Normalized PSD plots

\[ S_u(f) / A^2 T \]

- Blue: Unipolar NRZ
- Red: Polar NRZ
- Black: Manchester

\[ fT \]
Bipolar NRZ

• Successive 1’s have alternating polarity

\[ 0 \rightarrow \text{Zero amplitude} \]
\[ 1 \rightarrow +A \text{ or } -A \]

• Probability mass function of \( b[n] \)

\[
    P (b[n] = 0) = \frac{1}{2}
\]
\[
    P (b[n] = -A) = \frac{1}{4}
\]
\[
    P (b[n] = A) = \frac{1}{4}
\]

• Symbols are identically distributed but they are not independent
Bipolar NRZ

- Autocorrelation of $b[n]$ sequence

$$R_b[k] = \begin{cases} 
A^2/2 & k = 0 \\
-A^2/4 & k = \pm 1 \\
0 & \text{otherwise}
\end{cases}$$

- Power Spectral Density

$$S_u(f) = T \text{sinc}^2(fT) \left[ \frac{A^2}{2} - \frac{A^2}{4} \left( e^{j2\pi fT} + e^{-j2\pi fT} \right) \right]$$

$$= \frac{A^2 T}{2} \text{sinc}^2(fT) \left[ 1 - \cos(2\pi fT) \right]$$

$$= A^2 T \text{sinc}^2(fT) \sin^2(\pi fT)$$
Thanks for your attention