# Performance of ML Receiver for *M*-ary Signaling

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# Performance of ML Decision Rule for *M*-ary signaling

# ML Decision Rule for $M$-ary Signaling

- $M$ equally likely hypotheses

$$\begin{array}{ccc} H_1 & : & y(t) = s_1(t) + n(t) \\ H_2 & : & y(t) = s_2(t) + n(t) \\ \vdots & & \vdots \\ H_M & : & y(t) = s_M(t) + n(t) \end{array}$$

- The ML decision rule for real AWGN channel is

$$\delta_{ML}(y) = \underset{1 \le i \le M}{\text{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le M}{\text{argmax}} \langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$$

- The ML decision rule for complex AWGN channel is

$$\delta_{ML}(y) = \underset{1 \le i \le M}{\text{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le M}{\text{argmax}} \, \text{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}$$

- In general, there is no neat expression for $P_e$ as in the binary case

# QPSK

- QPSK signals where $p(t)$ is a real baseband pulse of duration $T$

$$\begin{aligned} s_1^p(t) &= \sqrt{2}p(t)\cos\left(2\pi f_c t + \frac{\pi}{4}\right) \\ s_2^p(t) &= \sqrt{2}p(t)\cos\left(2\pi f_c t + \frac{3\pi}{4}\right) \\ s_3^p(t) &= \sqrt{2}p(t)\cos\left(2\pi f_c t + \frac{5\pi}{4}\right) \\ s_4^p(t) &= \sqrt{2}p(t)\cos\left(2\pi f_c t + \frac{7\pi}{4}\right) \end{aligned}$$

- Complex envelopes of QPSK Signals

$$s_1(t) = p(t)e^{j\frac{\pi}{4}}, s_2(t) = p(t)e^{j\frac{3\pi}{4}}, s_3(t) = p(t)e^{j\frac{5\pi}{4}}, s_4(t) = p(t)e^{j\frac{7\pi}{4}}$$

- Orthonormal basis for the complex envelopes consists of only

$$\phi(t) = \frac{p(t)}{\sqrt{E_p}}$$

# ML Receiver for QPSK

- $E_b = E_p/2$
- The vector representation of the QPSK signals is

$$\begin{aligned} s_1 &= \sqrt{E_b} + j\sqrt{E_b} \\ s_2 &= -\sqrt{E_b} + j\sqrt{E_b} \\ s_3 &= -\sqrt{E_b} - j\sqrt{E_b} \\ s_4 &= \sqrt{E_b} - j\sqrt{E_b} \end{aligned}$$

- The hypothesis testing problem in terms of vectors is

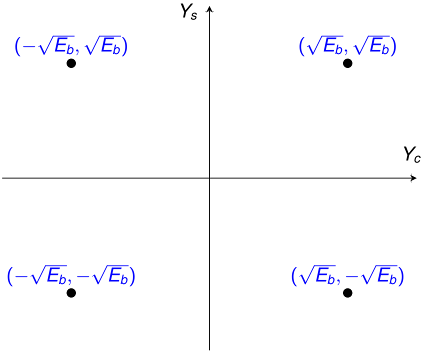
$$H_i : Y = s_i + N, \quad i = 1, \dots, 4$$

  where $N \sim \mathcal{CN}(0, 2\sigma^2)$
- The ML decision rule is given by

$$\delta_{ML}(y) = \underset{1 \le i \le 4}{\operatorname{argmin}} \|y - s_i\|^2 = \underset{1 \le i \le 4}{\operatorname{argmax}} \operatorname{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}$$

- The ML decision rule decides $s_i$ was transmitted if $y$ belongs to the $i$th quadrant

# ML Decision Rule for QPSK




$$P_{e|1} = \Pr\left[ Y_c < 0 \text{ or } Y_s < 0 \middle| (\sqrt{E_b}, \sqrt{E_b}) \text{ was sent} \right]$$

# ML Decision Rule for QPSK

- Probability of error when $s_1$ is transmitted is

$$\begin{aligned} P_{e|1} &= \Pr\left[ Y_c < 0 \text{ or } Y_s < 0 \middle| (\sqrt{E_b}, \sqrt{E_b}) \text{ was sent} \right] \\ &= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right) \end{aligned}$$

- By symmetry,

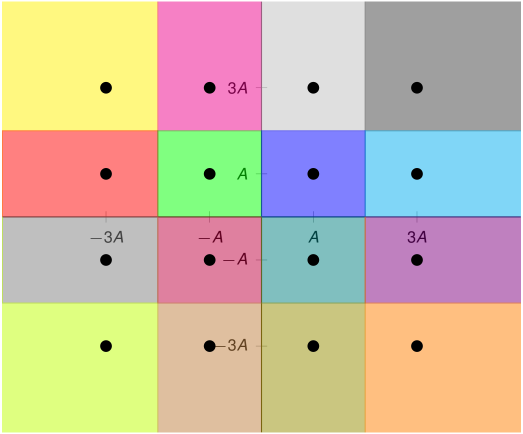
$$P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$$

- The average probability of error is

$$P_e = \frac{1}{4}\sum_{i=1}^{4} P_{e|i} = P_{e|1} = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

# ML Decision Rule for 16-QAM

16-QAM



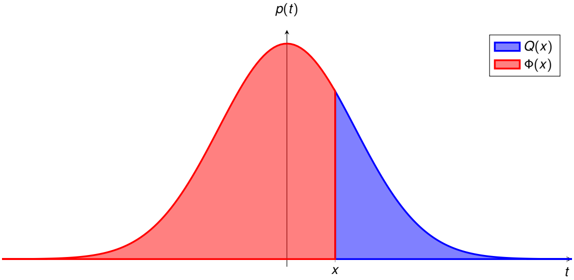

Exact analysis is tedious. Approximate analysis is sufficient.

# Revisiting the $Q$ function

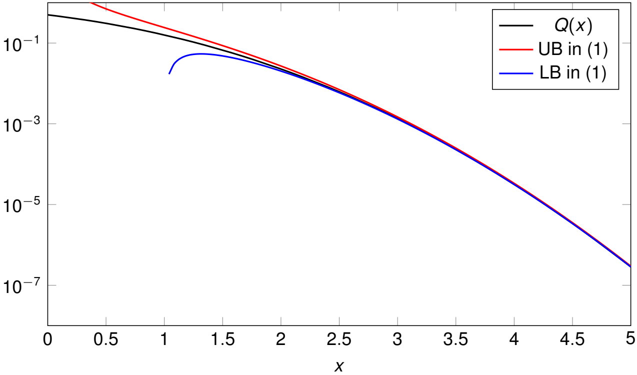
# Revisiting the $Q$ function

$X \sim N(0, 1)$

$$Q(x) = P\left[X > x\right] = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

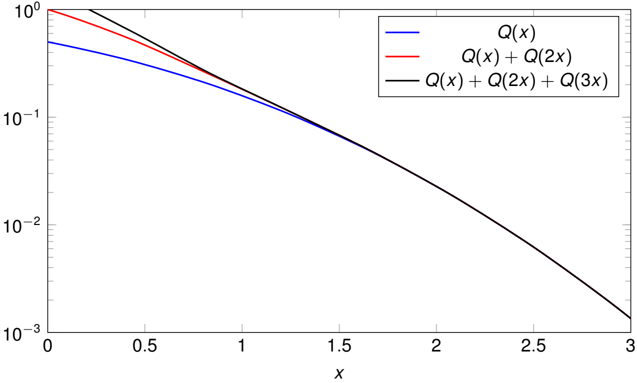


# Bounds on $Q(x)$ for Large Arguments




$$\left(1 - \frac{1}{x^2}\right) \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \leq Q(x) \leq \frac{e^{-\frac{x^2}{2}}}{x\sqrt{2\pi}} \tag{1}$$

# $Q$ Functions with Smallest Arguments Dominate




- $P_e$ in AWGN channels can be bounded by a sum of $Q$ functions
- The $Q$ function with the smallest argument is used to approximate $P_e$

# Union Bound

# Union Bound for $M$-ary Signaling in AWGN

- Let $Z_i$ be $\langle y, s_i \rangle - \frac{\|s_i\|^2}{2}$ or $\text{Re}\left(\langle y, s_i \rangle\right) - \frac{\|s_i\|^2}{2}$
- The conditional error probability given $H_i$ is true is

$$P_{e|i} = \Pr\left[\bigcup_{j \neq i} \{Z_i < Z_j\} \middle| H_i\right]$$

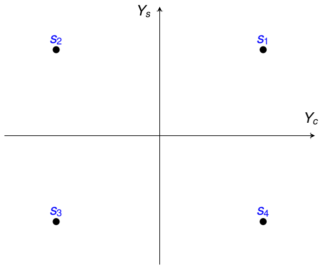
- Since $P(A \cup B) \leq P(A) + P(B)$, we have

$$P_{e|i} \leq \sum_{j \neq i} \Pr\left[Z_i < Z_j \middle| H_i\right] = \sum_{j \neq i} Q\left(\frac{\|s_j - s_i\|}{2\sigma}\right)$$

- The error probability is given by

$$P_e = \frac{1}{M}\sum_{i=1}^{M} P_{e|i} \leq \frac{1}{M}\sum_{i=1}^{M}\sum_{j \neq i} Q\left(\frac{\|s_j - s_i\|}{2\sigma}\right)$$

# Union Bound for QPSK




$$P_{e|1} = \Pr\left[\cup_{j\neq 1}\{Z_1 < Z_j\}\middle| H_1\right] \leq \sum_{j\neq 1} \Pr\left[Z_1 < Z_j \middle| H_1\right]$$

$$P_{e|1} \leq Q\left(\frac{\|s_2 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_3 - s_1\|}{2\sigma}\right) + Q\left(\frac{\|s_4 - s_1\|}{2\sigma}\right)$$

$$= 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

# Union Bound for QPSK

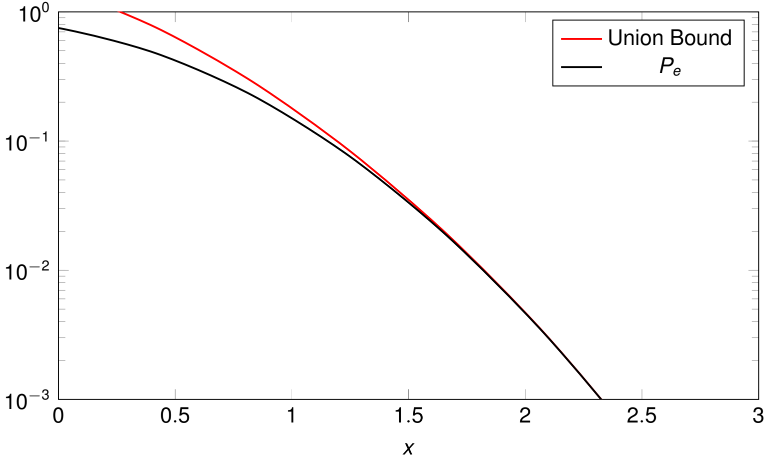
- Union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

- Exact error probability of ML rule

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

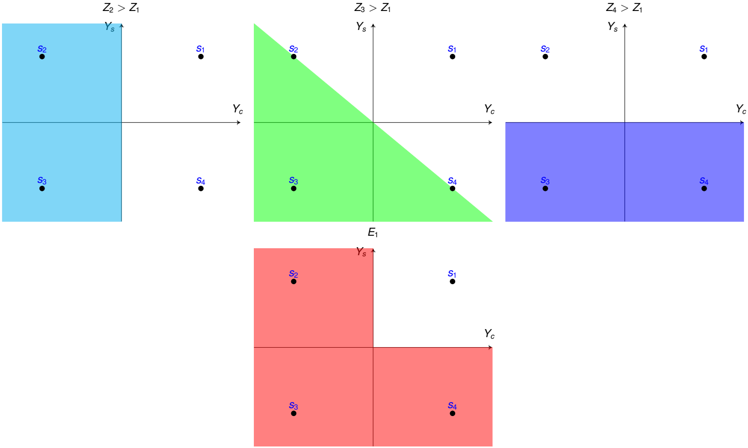
# Union Bound and Exact Error Probability for QPSK



# Intelligent Union Bound

# QPSK Error Events

$$E_1 = [Z_2 > Z_1] \cup [Z_3 > Z_1] \cup [Z_4 > Z_1] = [Z_2 > Z_1] \cup [Z_4 > Z_1]$$



# Intelligent Union Bound for QPSK

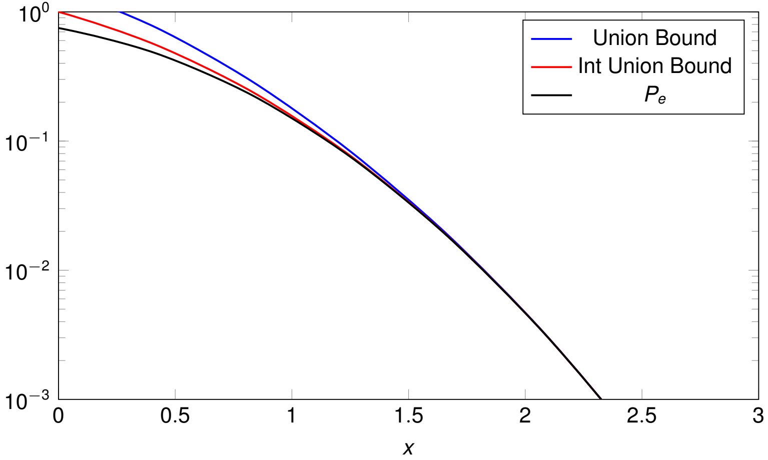
- Intelligent union bound on $P_{e|1}$

$$
\begin{aligned}
P_{e|1} &= \Pr\left[ (Z_2 > Z_1) \cup (Z_4 > Z_1) \middle| H_1 \right] \\
&\le \Pr\left[ Z_2 > Z_1 \middle| H_1 \right] + \Pr\left[ Z_4 > Z_1 \middle| H_1 \right] \\
&= Q\left( \frac{\|s_2 - s_1\|}{2\sigma} \right) + Q\left( \frac{\|s_4 - s_1\|}{2\sigma} \right) \\
&= 2Q\left( \sqrt{\frac{2E_b}{N_0}} \right)
\end{aligned}
$$

- By symmetry $P_{e|1} = P_{e|2} = P_{e|3} = P_{e|4}$ and

$$
P_e \le 2Q\left( \sqrt{\frac{2E_b}{N_0}} \right)
$$

# Intelligent Union Bound and Exact Error Probability for QPSK



# General Strategy for Intelligent Union Bound

- Let $N_{ML}(i)$ be the smallest set of neighbors of $s_i$ which define the decision region $\Gamma_i$

$$\Gamma_i = \left\{ y \middle| \delta_{ML}(y) = i \right\} = \left\{ y \middle| Z_i \geq Z_j \text{ for all } j \in N_{ML}(i) \right\}$$

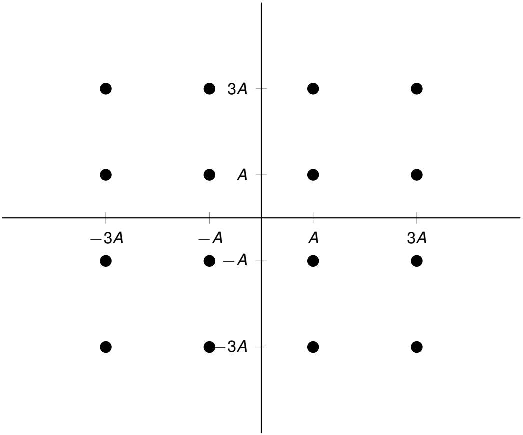
- Probability of error when $s_i$ is transmitted is

$$\begin{aligned} P_{e|i} &= \Pr\left[y \notin \Gamma_i | H_i\right] = \Pr\left[ Z_i < Z_j \text{ for some } j \in N_{ML}(i) \middle| H_i \right] \\ &\leq \sum_{j \in N_{ML}(i)} Q\left( \frac{\|s_j - s_i\|}{2\sigma} \right) \end{aligned}$$

- Average probability of error is bounded by

$$P_e \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{j \in N_{ML}(i)} Q\left( \frac{\|s_j - s_i\|}{2\sigma} \right)$$

# Intelligent Union Bound for 16-QAM




Assignment 5

# Nearest Neighbors Approximation

# Nearest Neighbors Approximation

- Let $d_{min}$ be the minimum distance between constellation points

$$d_{min} = \min_{i \neq j} \|s_i - s_j\|$$

- Let $N_{d_{min}}(i)$ denote the number of nearest neighbors of $s_i$

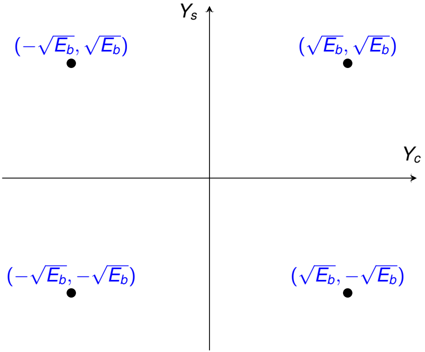
$$P_{e|i} \approx N_{d_{min}}(i) Q\left(\frac{d_{min}}{2\sigma}\right)$$

- Averaging over $i$ we get

$$P_e \approx \bar{N}_{d_{min}} Q\left(\frac{d_{min}}{2\sigma}\right)$$

where $\bar{N}_{d_{min}}$ denotes the average number of nearest neighbors

# Nearest Neighbors Approximation for QPSK




$$d_{min} = 2\sqrt{E_b}, \qquad N_{d_{min}}(1) = 2 = \bar{N}_{d_{min}}$$

$$P_e \approx \bar{N}_{d_{min}} Q\left(\frac{d_{min}}{2\sigma}\right) = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

# Summary of results for QPSK

- Exact error probability of ML rule

$$P_e = 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) - Q^2\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right) + Q\left(\sqrt{\frac{4E_b}{N_0}}\right)$$

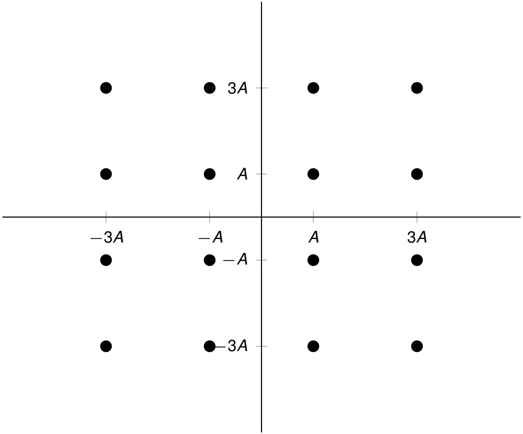
- Intelligent union bound on error probability of ML rule

$$P_e \leq 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

- Nearest neighbors approximation of error probability of ML rule

$$P_e \approx 2Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

# Nearest Neighbors Approximation for 16-QAM




Assignment 5

Thanks for your attention