EE 703: Digital Message Transmission (Autumn 2016)
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1. A probability measure is a mapping P on the event space which satisfies
   (i) $0 \leq P(E) \leq 1$
   (ii) $P(\Omega) = 1$
   (iii) For any sequence of events $E_1, E_2, \ldots$ that are pairwise mutually exclusive, i.e. $E_n \cap E_m = \phi$ for $n \neq m$,
   $$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$
   Using these properties, show that
   (a) $P(A^c) = 1 - P(A)$
   (b) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2. Let Y and Z be independent random variables such that Z is equally likely to be 1 or -1 and Y is equally likely to be 1 or 2. Let $X = YZ$. Prove that X and Y are uncorrelated but not independent.

3. Suppose $A_i, 1 \leq i \leq 5$, are independent events. Show that
   (a) $(A_1 \cup A_2) \cap A_3$ and $A_4^c \cup A_5^c$ are independent.
   (b) $(A_1 \cup A_2)$, A_3 and A_5^c are independent. (Note: There are three events here)

4. Using the clues given below, fill in the missing entries in the joint probability mass function of X and Y.

Y/X	1	2	3
1	?	?	?
2	?	0	?
3	0	?	0

Table 1: Joint probability mass function $f_{X,Y}(x, y)$

   For $k = 1, 2, 3$,
   - $P(Y = 1|X = k) = \frac{2}{3}$
   - $P(X = k|Y = 1) = \frac{k}{6}$

5. Suppose X and Y take values in $\{0, 1\}$ with joint probability mass function $f(x, y)$. Let $f(0, 0) = a, f(0, 1) = b, f(1, 0) = c$ and $f(1, 1) = d$. Find necessary and sufficient conditions for X and Y to be:
   (a) uncorrelated
   (b) independent

6. Let X and Y have joint probability density function $f(x, y) = 2e^{-x-y}$, $0 < x < y < \infty$. Find the expected values of X and Y.

7. Let X_n for $n \in \mathbb{Z}$ be a random process consisting of independent and identically distributed random variables. Show that X_n is a strict-sense stationary random process.

8. Let X_n for $n \in \mathbb{Z}$ be a wide-sense stationary random process with zero mean function and autocorrelation function given by

$$R_X[k] = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y_n = \frac{X_{n-1}+X_n+X_{n+1}}{3}$ be a filtered version of X_n. Calculate the autocorrelation function of Y_n.

9. Let $X(t) = \cos(2\pi f t + \Theta)$ be a random process where $\Theta \sim U[-\pi, \pi]$, i.e. Θ is uniformly distributed in $[-\pi, \pi]$. Calculate the mean function $\mu_X(t)$ and autocorrelation function $R_X(t_1, t_2)$ of $X(t)$.

10. Let $X_n = Z_1 + \cdots + Z_n$, $n = 1, 2, \ldots$ be a random process where Z_i are independent and identically distributed random variables with zero mean and variance σ^2. Calculate the mean function $\mu_X(n)$ and autocorrelation function $R_X(n_1, n_2)$ of X_n.

11. Two random vectors $\mathbf{X} = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_m \end{bmatrix}$ are independent if their joint probability density or mass function is a product of the marginal density or mass functions i.e. $p(\mathbf{X}, \mathbf{Y}) = p(\mathbf{X})p(\mathbf{Y})$.

    Two random processes $X(t)$ and $Y(t)$ are independent if any two vectors of time samples are independent i.e. $\begin{bmatrix} X(t_1) & X(t_2) & \cdots & X(t_n) \end{bmatrix}$ and $\begin{bmatrix} Y(\tau_1) & Y(\tau_2) & \cdots & Y(\tau_m) \end{bmatrix}$ are independent vectors as per the previous definition for any $n, m \in \mathbb{N}$ and any $t_1, t_2, \ldots, t_n, \tau_1, \tau_2, \ldots, \tau_m \in \mathbb{R}$.

    Suppose $X(t)$ and $Y(t)$ are independent wide-sense stationary random processes with mean functions equal to μ_X and μ_Y respectively. Let their autocorrelation functions be $R_X(\tau)$ and $R_Y(\tau)$ respectively.

    (a) Show that $Z(t) = X(t) + Y(t)$ is a wide-sense stationary random process.

    (b) Show that $W(t) = X(t)Y(t)$ is a wide-sense stationary random process.