EE 703: Digital Message Transmission (Autumn 2016)
Instructor: Saravanan Vijayakumaran
Indian Institute of Technology Bombay

Problem Set 3 — Date: November 2, 2016

1. Prove that $\mathbf{U} = e^{j\phi}\mathbf{Z}$ is a complex Gaussian vector when $\mathbf{Z}$ is a complex Gaussian vector.

2. Consider the following signals.

$$s_1(t) = -3Ap(t), s_2(t) = -Ap(t), s_3(t) = Ap(t), s_4(t) = 3Ap(t)$$

where $p(t) = I_{[0,1]}(t)$. If these signals are equally likely to be sent over a real AWGN channel with power spectral density $\frac{N_0}{2}$, derive the following when the optimal receiver is used.

   (a) The power efficiency of this modulation scheme.

   (b) The exact symbol error probability as a function of E_b and N_0

   (c) The union bound on the symbol error probability

   (d) The intelligent union bound on the symbol error probability

   (e) The nearest neighbor approximation of the symbol error probability

3. For $M = 2^b$, suppose M **orthogonal** real signals $s_i(t)$, $i = 1, \ldots, M$ are used for transmitting b bits over a real AWGN channel with PSD $\frac{N_0}{2}$. If all the signals have the same energy E and are equally likely to be transmitted, derive the following as a function of E, N_0, b or M when the optimal receiver is used.

   (a) The power efficiency of this modulation scheme

   (b) The union bound on the symbol error probability

   (c) The nearest neighbor approximation of the symbol error probability

4. Suppose observations Y_i, $i = 1, 2, \ldots, N$ are Poisson distributed with parameter λ. Assume that the Y_i's are independent.

   (a) Derive the ML estimator for λ.

   (b) Find the mean and variance of the ML estimate.

   Recall that a Poisson distributed random variable with parameter λ has a probability mass function given by

$$\Pr(Y = n) = \frac{e^{-\lambda}\lambda^n}{n!}, n = 0, 1, 2, \ldots$$

   with mean and variance both equal to λ.

5. Suppose observations X_i and Y_i $(i = 1, \ldots, N)$ depend on an unknown parameter A as per the following distributions.

$$X_i \sim \mathcal{N}(A, \sigma^2), \quad i = 1, 2, \ldots, N$$
$$Y_i \sim \mathcal{N}(A, 2\sigma^2), \quad i = 1, 2, \ldots, N$$

   Note that the variance of Y_i is twice the variance of X_i. Assume that X_i and X_j are independent for $i \neq j$. Assume that Y_i and Y_j are independent for $i \neq j$. Assume that X_i and Y_j are independent for all i, j. Assume σ^2 is known.

   (a) the ML estimator for A.

(b) Find the mean and variance of the ML estimate.

6. Suppose we observe a sequence of real values $Y_1, Y_2, \ldots, Y_n$ given by

$$Y_k = \theta s_k + N_k, \quad k = 1, 2, \ldots, n$$

where $\mathbf{N} = \begin{bmatrix} N_1 & N_2 & \cdots & N_n \end{bmatrix}^T$ is a zero-mean Gaussian vector with known covariance matrix $\mathbf{\Sigma}$ which is a positive definite matrix. The sequence $s_1, \ldots, s_n$ is a known signal sequence and θ is an unknown parameter.

(a) Find the ML estimate $\hat{\theta}_{ML}(\mathbf{Y})$ of the parameter θ.

(b) Find the mean and variance of $\hat{\theta}_{ML}(\mathbf{Y})$.

Recall that the pdf of a real $n \times 1$ Gaussian vector $\mathbf{x}$ with mean vector $\mathbf{m}$ and covariance matrix $\mathbf{C}$ is given by

$$p(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{C})}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})\right)$$