EE 703: Digital Message Transmission (Autumn 2016)
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1. (5 points) Let $u_p(t)$ and $v_p(t)$ be real passband signals with complex baseband representations $u(t)$ and $v(t)$ respectively. Let $u(t) = u_c(t) + ju_s(t)$ and $v(t) = v_c(t) + jv_s(t)$.
   (a) Show that $\langle u_p, v_p \rangle = \langle u_c, v_c \rangle + \langle u_s, v_s \rangle$.
   (b) Show that $\|u_p\| = \|u\|$.

2. (5 points) Let $\phi_1(t), \phi_2(t), \ldots, \phi_N(t)$ be an orthonormal basis for a set of real signals $s_1(t)$, $s_2(t), \ldots, s_M(t)$. Let $\mathbf{s}_i \in \mathbb{R}^N$ be the signal space representation of $s_i(t)$. Show that $\langle s_i, s_j \rangle = \langle \mathbf{s}_i, \mathbf{s}_j \rangle$.

3. (5 points) Let $X(t)$ be a linearly modulated signal given by

$$X(t) = \sum_{n=-\infty}^{\infty} a_n s(t - nT)$$

   where $a_n$ is a wide-sense stationary random process representing the symbols and $s(t)$ is a pulse of duration $T$. Derive the power spectral density of $X(t)$.

4. (5 points) Let $n(t)$ be a real white Gaussian noise process with PSD $S_n(f) = \sigma^2$. Let $u_1(t)$ and $u_2(t)$ be two non-zero deterministic finite energy signals which are orthogonal, i.e. $\langle u_1, u_2 \rangle = 0$. Show that $\langle n, u_1 \rangle$ and $\langle n, u_2 \rangle$ are independent random variables.

5. (5 points) Consider the binary hypothesis testing problem below where the hypothesis are equally likely and $\sigma_1 > \sigma_0$.

$$H_0 : Y \sim \mathcal{N}(0, \sigma_0^2)$$
$$H_1 : Y \sim \mathcal{N}(0, \sigma_1^2)$$

   (a) Derive the optimal decision for this hypothesis testing problem. Simplify it as much as possible.
   (b) Derive the average decision error probability $P_e$ of this optimal decision rule.

6. (5 points) Consider the binary hypothesis testing problem below where the hypothesis are equally likely, $s_0(t)$ and $s_1(t)$ are distinct finite energy signals, and $n(t)$ is a real white Gaussian noise process with PSD $\sigma^2$.

$$H_0 : y(t) = s_0(t) + n(t)$$
$$H_1 : y(t) = s_1(t) + n(t)$$

   (a) Derive the optimal decision for this hypothesis testing problem. Simplify it to the form of "a single random variable being compared to a threshold".
   (b) Derive the average decision error probability $P_e$ of this optimal decision rule.
   (c) Show that $P_e$ is minimized when $s_0(t)$ and $s_1(t)$ are antipodal signals.

7. (5 points) A binomial random variable $X$ with parameters $N$ and $p$ takes values $0, 1, 2, \ldots, N$ and has probability mass function given by

$$\Pr[X = k] = \binom{N}{k} p^k (1-p)^{N-k}, \quad \text{for } k = 0, 1, 2, \ldots, N.$$

   Assume that the integer parameter $N$ is known. Find the ML estimate of $p$ given that you observe a single instance of the random variable $X$.

8. (10 points) For the constellation of 8 symbols, assume that the transmitted symbol is corrupted by adding $N \sim \mathcal{CN}(0, N_0)$. If all the constellation points are equally likely to be transmitted, calculate the BER performance of the ML receiver under Gray mapping in terms of $E_b$ and $N_0$.