EE 703: Digital Message Transmission (Autumn 2020)
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1. [5 points] Consider
$$u(t) = \sum_{n=-\infty}^{\infty} b_n p(t - nT)$$
where $p(t) = I_{[0,T)}(t)$ for a fixed time duration $T > 0$. Recall that $I_A(t)$ is the indicator function of the set A.
   (a) Prove that $u(t)$ is a cyclostationary random process with respect to period T if $\{b_n\}$ is a discrete-time stationary random process.
   (b) Prove that $u(t)$ is a wide-sense cyclostationary random process with respect to period T if $\{b_n\}$ is a discrete-time wide-sense stationary random process.

2. [5 points] Consider the following M-ary hypothesis testing problem:
$$\begin{aligned} H_1 &: y(t) = s_1(t) + n(t) \\ H_2 &: y(t) = s_2(t) + n(t) \\ &\vdots \\ H_M &: y(t) = s_M(t) + n(t) \end{aligned}$$
where $s_1(t), s_2(t), \ldots, s_M(t)$ are real signals which are nonzero for $-T \leq t \leq T$ for a fixed $T > 0$ and $n(t)$ is white Gaussian noise (also real). The hypothesis H_i occurs with probability π_i where $\sum_{i=1}^{M} \pi_i = 1$.

   We are given that
$$s_i(t) = s_j(t) \text{ for } t \in [-T, 0]$$
for all $i, j \in \{1, 2, \ldots, M\}$. Show that the optimal receiver can ignore the signal received in the interval $[-T, 0]$ in taking its decision.

3. [5 points] Consider the binary hypothesis testing problem below where the hypothesis are equally likely, $s_0(t)$ and $s_1(t)$ are distinct finite energy real signals, and $n(t)$ is a real white Gaussian noise process with PSD σ^2.
$$\begin{aligned} H_0 &: y(t) = s_0(t) + n(t), \\ H_1 &: y(t) = s_1(t) + n(t). \end{aligned}$$
   (a) Derive the optimal decision rule for this hypothesis testing problem. You can use the optimal decision rule for M-ary hypothesis testing without giving a proof. Simplify it to the form of "a single random variable being compared to a threshold".
   (b) Derive the average decision error probability P_e of this optimal decision rule.
   (c) Show that P_e is minimized when $s_0(t)$ and $s_1(t)$ are antipodal signals, that is when $s_0(t) = -s_1(t)$.

4. (a) [2 points] Let $b \geq 1$ be an integer. For $M = 2^b$, suppose M **orthogonal** real signals $s_i(t)$, $i = 1, \ldots, M$ are used for transmitting b bits over a real AWGN channel with PSD $\frac{N_0}{2}$. If all the signals have the same energy E and are equally likely to be transmitted, derive the following as a function of E, N_0, b or M when the optimal receiver is used.

i. The union bound on the symbol error probability

ii. The nearest neighbor approximation of the symbol error probability

(b) [3 points] Suppose we use the M signals in the previous part to form a set of $2M$ real signals

$$\{s_1(t), s_2(t), \dots, s_M(t), -s_1(t), -s_2(t), \dots, -s_M(t)\}.$$

So the set contains M signals and their negative versions. These $2M$ signals are used for transmitting $b+1$ bits over a real AWGN channel with PSD $\frac{N_0}{2}$. If all the $2M$ signals are equally likely to be transmitted, derive the following as a function of E, N_0, b or M when the optimal receiver is used.

i. The union bound on the symbol error probability

ii. The nearest neighbor approximation of the symbol error probability

5. [5 points] Consider the M-ary hypothesis testing problem in AWGN where $s_i(t) = A_i p(t)$ for a unit energy real signal $p(t)$ which is nonzero for $0 \le t \le T$ and $A_i \in \mathbb{R}$ such that $A_1 > A_2 > \cdots > A_M$.

$$\begin{aligned} H_1 &: \quad y(t) = s_1(t) + n(t) \\ H_2 &: \quad y(t) = s_2(t) + n(t) \\ &\vdots \qquad\qquad \vdots \\ H_M &: \quad y(t) = s_M(t) + n(t) \end{aligned}$$

If all the hypotheses are equally likely, show that the optimal receiver compares the output of a matched filter to a set of thresholds. You can use the optimal decision rule for M-ary hypothesis testing without giving a proof.

6. [5 points] Consider the following binary hypothesis testing problem where the hypothesis are equally likely.

$$\begin{aligned} H_0 &: Y_i \sim \text{Bernoulli}(p_0) \text{ for } i = 1, 2, \dots, N, \\ H_1 &: Y_i \sim \text{Bernoulli}(p_1) \text{ for } i = 1, 2, \dots, N. \end{aligned}$$

Assume that the Y_i's are independent under both hypotheses. Assume that $p_1 > p_0$. Note that p_0 and p_1 are known.

(a) Derive the optimal decision rule.

(b) Find the decision error probability of the optimal decision rule.

7. [5 points] Suppose X is a Gaussian random variable with mean μ and variance $\sigma^2 > 0$. The parameters μ and σ^2 are known. We observe Y given by

$$Y = X + N,$$

where N is a Gaussian random variable with mean 0 and variance σ^2. Assume that X is independent of N. Find the MMSE estimator of X given Y.

8. [5 points] Suppose N_1, N_2 are independent Gaussian random variables each having mean 0 and variance $\sigma^2 > 0$. The variance σ^2 is assumed to be known. We observe two observations Y_1, Y_2 given by

$$\begin{aligned} Y_1 &= \lambda + N_1, \\ Y_2 &= 2\lambda + N_1 + N_2. \end{aligned}$$

(a) Find the ML estimator of the parameter λ.

(b) Perform a sanity check on the formula you got for the ML estimator. Explain why it makes sense.

(c) Find the mean and variance of the ML estimator.